

THE SHAPES OF MARKET AREAS AND WELFARE IN SPATIAL EQUILIBRIA

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I. INTRODUCTION

This article examines the economy of market areas for firms and impact on social surplus in that area. There exist a number of studies concerning the shapes of market areas in spatial competitive equilibrium, which are all based on Lösch's study [15]. Lösch analyzed the relationship between the triangular, square, or hexagonal market area and the level of demand, and concluded that optimal market area, i.e., which contains a given amount of demand with the smallest area is a hexagon. Mills and Lav [13] and Hartwick [10] studies the relationship between the shape of market areas and the profits of firms, and thereby extended the Lösch theory.

Greenhut and Ohta [8], on the other hand, used a spatial competitive model in contrast to Lösch to analyze the relationship between the shapes of market areas and the profits. Greenhut and Ohta considered the problem of market region shapes from a viewpoint of the economic continuum from a monopoly to a competitive equilibrium, and concluded that the optimal market area has to be hexagonal.

These studies, although based on different analytical frameworks, asserted in common that a market area in a competitive equilibrium is hexagonal, and argued the following point as the rationale or as a supple-

ment of their arguments: when the market area is hexagonal, the number of firms per unit of area is maximized. On the other hand, Eaton and Lipsey [7] investigated the shapes of market areas, under a precondition different from the aforementioned theoretical analyses, and showed that optimum shape of market area can be rectangle or other shapes as well as a hexagon.

An objective of this article is to demonstrate the following point : The existing studies with the assumption of costless relocation have so far asserted that the shape of market area that makes the number of firms per unit of area the largest be a hexagon. However, even under this assumption of costless relocation, that shape which makes the number of firms the largest can also be a triangle or a square, not to mention a hexagon. Which shape the market area takes will depend on the type of spatial competition and the level of production costs. These ideas will be examined in section IV.

The literature on the economic welfare in a spatial economy has been accumulating since Mills and Lav's analysis [13] of free entry in a spatial market and economic welfare. The studies by Beckmann [2], Benson [3], Denike and Parr [6], Holahan and Schuler [11, 12], and Ohta [16] are examples. However, analyses concerning the shape of market areas and level of welfare have hitherto been limited.¹⁾ Therefore, our second main objective is to analyze the relationship between the shape of market area and the social surplus in a spatial monopoly and a competitive equilibrium. We will demonstrate that in a monopolistic equilibrium, the social surplus per unit of area is maximized when a monopolistic firm has a triangular market area. On the other hand, in a competitive equilibrium, the shape that maximizes the social surplus per unit of area will depend on the type

of spatial competition and the level of production costs. The shape will be a square or a hexagon.

In the next section, our assumptions will be discussed and the profit functions of various shaped market areas will be derived. The shape of the market areas and social surplus in a monopolistic equilibrium will be analyzed in section III. Section IV will analyze the shape of market area and social surplus in a competitive equilibrium. The results will be summarized in the final section.

II . THE SHAPES OF MARKET AREAS AND THE PROFIT FUNCTIONS OF FIRMS

A. Assumptions

Our arguments will be made under the following assumptions.

- 1) Consumers are uniformly and continuously distributed on a plane.
- 2) The consumer demand function is ;

$$q = a - g(P + wu) \quad (1)$$

where q is the quantity demanded by the consumer, P is the price quoted at factory, u is the distance from a firm to the consumer, w is the freight rate, and a and g are positive constants (w and g are both assumed to be 1 in the following analysis).

- 3) All consumers are supplied with goods.
- 4) Transportation costs are born by the consumer.

1) Holahan and Schuler [11, 12] analyzed the welfare levels in several sizes of hexagonal and square market areas.

5) The firm's cost functions is:

$$TC = KQ + F \quad (2)$$

where TC is the total cost, K is the marginal cost, Q is the amount of sales, and F is the fixed cost.²⁾

6) Each firm can move its location without incurring costs.

Thus, we will, under the assumption of costless relocation, make an attempt at spatial equilibrium analysis within the framework of normative concepts.

B. THE PROFIT FUNCTION OF FIRMS

The market area of the firms satisfying the assumption 3) above must be a shape which the firm can completely cover with individual market areas of the same shape. Therefore, a circular market area, which necessarily leaves an unsupplied area, should be excluded. For this analysis, potential shapes for market areas will be confined to the triangle, the square, or the hexagon.³⁾ Now we will derive the profits for firms in these three shapes of market regions.

2) When competitive equilibrium is analyzed, from the standpoint of usual micro economics, fixed cost F is considered to be variable. However, we assume that all management fixed cost are not variable, but constant. The reason why we do this is the following: We also, following Mills-Lav [13] consider only static industry equilibria. That is, we make no attempt to ascertain whether any adjustment process will converge to industry equilibrium from any particular initial arrangement of firms and market area.

Profits are generally given by the following:

$$Y = (P - K)Q - F \quad (3)$$

where Y denotes profits of the firm. The amount of sales Q must be known in order to derive profits. Generally, the quantity demanded for a regular polygon market area with an inscribed circle of radius U is given by

$$Q = 2\rho \int_0^{\pi/\rho} \int_0^{U/\cos\theta} (a - (p+u)) u du d\theta \quad (4)$$

where ρ is the number of sides of the regular polygon, and θ is the center angle formed by the lines connecting the consumers located at a middle point of a side and the consumers located at a vertex of the market area with the firm located in the middle of that market area. In Figure 1, an example of a squared market area is shown, with $\rho = 4$ and $\theta = 45^\circ$.

3) This can be proved as follows. Suppose that a plane can be completely covered by the same shape of a regular polygon. This regular n -polygon has each vertex concentrated by other m vertexes. A vertex of the regular n -polygon is described as

$$(n-2)\pi/n \quad (F-1)$$

The condition that m times of a vertical angle equal to 2π is

$$m(1-2/n) = 2 \quad (F-2)$$

From this equation, $m \cdot n = 2m + 2n$. Adding 4 to both sides of this equation and transforming it, the equation (F-2) becomes

$$(m-2)(n-2) = 4 \quad (F-3)$$

Integer solutions for the equation (F-3) are three pairs of

$$(m, n) = (4, 4), (3, 6), (6, 3).$$

Hence, the shapes that completely cover a plane are three: a regular triangle, a regular square, or a regular hexagon (see Miyasaka [14]).

Now, from the equations (3) and (4) , the profits in three shapes of market area, Y_t , Y_s , and Y_h (t,s, and h denote a triangle, a square, and a hexagon, respectively) are given as follows:

$$Y_t = 6U_t^2(\sqrt{3}/2 \cdot a - \sqrt{3}/2 \cdot p_t - 0.7969U_t)(p_t - K) - F \quad (5t)$$

$$Y_s = 8U_s^2(1/2 \cdot a - 1/2 \cdot p_s - 0.3826U_s)(p_s - K) - F \quad (5s)$$

$$Y_h = 12U_h^2(1/2\sqrt{3} \cdot a - 1/2\sqrt{3} \cdot p_h - 0.2027U_h)(p_h - K) - F \quad (5h)$$

III. THE SHAPES OF MARKET AREAS IN MONOPOLISTIC EQUILIBRIUM AND SOCIAL SURPLUS

A. Equilibrium Values Under A Monopolistic Equilibrium

In this section let us suppose the following case. Namely, following assumption 3) , the market area of each firm is limited to a triangle, a square, or to a hexagon. A plane must be completely covered by a large number of the equal shaped market areas. Under these conditions each firm can, without competition among each others, determine the price and the size of market area, as a monopoly-firm can.⁴⁾ Let us calculate for the aforementioned three different shaped market areas, assuming these adjustments can be made, the firm's mill price, profit, and the length of a radius of the inscribed circle of market areas. A firm determines the mill price and the size of marketing area so that it will maximize profits. These

4) In the case of monopoly management, the management's market area becomes a circle. The reason why we analyze the shapes of market areas under monopoly management, on the assumption of a triangle, a square and a hexagon is that we want to compare the results from spatial competitive equilibrium with the results from monopolistic equilibrium.

values are determined as follows.

Let the distance from a firm to a vertex of the market area be $\bar{U}$. In a monopoly, the quantity demanded becomes zero at a point located $\bar{U}$ units of distance from the firm. Thus, the equation (1) becomes

$$a - (P + \bar{U}) = 0 \quad (6)$$

From the equation (6), $\bar{U} = a - P$. Since $U_t = 1/2 \cdot \bar{U}_t$, $U_s = 1/\sqrt{2} \cdot \bar{U}_s$,

$U_h = \sqrt{3}/2 \cdot \bar{U}_h$, U_i ($i = t, s, h$) can also be shown as

$$U_i = r_i \cdot (a - P_i) \quad (7)$$

$$i = t, s, h, r_t = 1/2, r_s = 1/\sqrt{2}, r_h = \sqrt{3}/2.$$

Using the equation (7), $P_i = a - U_i / r_i$. Putting this P_i into each P_i in the equation (5), each profit Y_i can be given as a function of P_i . The firm will determine P_i so that it will maximize its profits. The equilibrium mill price, P_i^* ($i = t, s, h$), can be obtained through differentiating the profit function by P_i , and equating the result to zero. That is,

$$dY_i/dP_i = 0 \quad i = t, s, h. \quad (8)$$

Once P_i^* is found, the radius, U_i^* of an inscribed circle of each market area in a monopoly can also be easily derived from equation (7). Table 1 lists the mill price P_i^* , the radius U_i^* of an inscribed circle and the profits Y_i under a monopolistic equilibrium. In Table 1, without loss of generality, the marginal cost K is, for simplicity, assumed to be zero. It is shown in Table 1 that the market area for a profit-maximizing firm becomes a hexagon, and the mill price is determined to be $0.25a$.

Table 1. The Equilibrium Values under a Monopolistic Equilibrium

		Triangle	Square	Hexagon
Mill Price,	P^*	0.25a	0.25a	0.25a
Radius of Inscribed Circle,	U^*	0.3750a	0.5303a	0.6495A
Profits,	Y	$0.074a^4 - F$	$0.0961a^4 - F$	$0.1074a^4 - F$

B. The Shapes of Market Areas and Social Surplus

We calculate for each shaped market area the values of social surplus per unit of area. The social value is defined as the sum of the consumer surplus and the producer surplus.⁵⁾ First, let us derive the consumer surplus per unit of area. The consumer surplus for each consumer can be represented by the area of a triangle formed by a demand curve, the price axis, and a line indicating the delivery price. Since the consumer surplus in a market region is a summation of the consumer surplus of each consumer, that value can be obtained in the following.

5) There is a case in which social welfare is considered as the sum of the consumer surplus and profit (Beckmann [2], p.51). However, in this paper, according to usual microeconomics, social welfare is defined as the sum of the consumer surplus and producer surplus.

$$CS_i = 2\rho \int_0^{\pi/\rho} \int_0^{U/\cos\theta} 1/2(a-p-u)^2 u du d\theta \quad (9)$$

$$= \alpha_i U_i^4 - \beta_i (a - P_i) U_i^3 + \gamma_i (a - P_i)^2 U_i^2 \quad (9')$$

$$i = t, s, h$$

$$\alpha_t = 2.5981, \alpha_s = 1.3333, \alpha_h = 0.9622,$$

$$\beta_t = 4.7811, \beta_s = 3.0607, \beta_h = 2.4320,$$

$$\gamma_t = 2.5981, \gamma_s = 2, \gamma_h = 1.7321,$$

where CS_i ($i = t, s, h$) denotes the consumer surplus in a market area. Substituting the corresponding figures in Table 1 for P_i and U_i in the equation (9'), the consumer surplus generated in each shape of market areas when firms are in the state of a monopoly can be calculated. The results of these calculation on the consumer surplus are shown in Table 2 for each case. It is clearly shown in table 2 that a hexagonal market area maximizes the consumer surplus.

Table 2. The Consumer Surplus under a Monopolistic Equilibrium

	Triangle	Square	Hexagon
Consumer Surplus	$0.0678a^4$	$0.0795a^4$	$0.0825a^4$

Next, let us compute the consumer surplus, CS_i^* ($i = t, s, h$), per unit of area. These can be computed by dividing the consumer surplus in each market shape by the total area of that same market area. Namely,

$$CS_i^* = CS_i / A_i U_i^2$$

$$i = t, s, h,$$

$$A_t = 5.1962, A_s = 4, A_h = 3.4641. \quad (10)$$

Using the equation (10), the consumer surplus per unit area of each shape of market area can be calculated.

If we assume that the marginal cost is zero, the value of producer surplus equates the revenue. Thus, the producer surplus per unit of area for each shape of market area can be obtained through deviding the revenue by the area of market region in question.

In short, the social surplus per unit of area can be calculated by adding the aforementioned consumer surplus and the producer surplus. The results are shown in Table 3. Table 3 indicates the following.

Table 3. Social Welfare per unit area under a Monopolistic Equilibrium

Triangle	Square	Hexagon
$0.194056a^2$	$0.1561a^2$	$0.12995a^2$

A triangular market region makes the social surplus per unit area the largest. We can conclude from this result that from the social viewpoint, when firms are in the state of a kind of monopoly described above, a triangle is the preferred shape of market area.

IV. THE SHAPES OF MARKET AREAS IN COMPETITIVE EQUILIBRIUM AND SOCIAL SURPLUS

The representative theoretical models on spatial competition are the Lösch type, the Greenhut=Ohta type, and the Hotelling and Smithies type model, as is well known. Although these three types of model does, as

Cappozza and Van Order [4] suggest, have some characteristics similar to the other two types of model.

When the values indicated below are substituted into the estimated variation against the radius U of the inscribed circle of market areas in the Hotelling-Smithies type, the same values (that is, concerning the price and the radius of the inscribed circle) as the competitive equilibria values in the Lösch and Greenhut-Ohta type model can be obtained. Hence our analysis will proceed within the framework of a Hotelling-Smithies type model.

A. Equilibrium Values in Spatial Competitive Equilibrium

In order for a spatial competitive equilibrium to be established, the following two conditions have to be simultaneously satisfied.⁶⁾

(A) a zero-profit condition

(B) a profit-maximization

The condition (A) means that as long as profits can be obtained in a market area, new firms enter the market so that the profit of every firm is eventually reduced to zero. On the other hand, the condition (B) implies that the firm determines the mill price and the size of market area so that profits are maximized. The condition (A) can be represented by equating the equation (5i) ($i = t, s, h$) to zero. Namely,

$$Y_i = 0 \quad (11)$$

$$i = t, s, h$$

The condition (B) can be obtained through partially differentiating the equation (5i) by the mill price P_i and equating the result to zero:

6) These two conditions are always required in the analyses based on the Lösch's study.

$$\begin{aligned}
\partial Y_i / \partial P_i &= [(\alpha_i U_i - \beta_i U_i P_i - \gamma_i U_i^2) \\
&\quad \times (P_i - K)(dU_i / dP_i) \\
&\quad + (1/2 \cdot \alpha_i U_i^2 - P_i \beta_i U_i^2 - 1/3 \cdot \gamma_i U_i^3 \\
&\quad + 1/2 \cdot K \beta_i U_i^2)] \cdot 2\rho_i \\
&= 0
\end{aligned} \tag{12}$$

$i = t, s, h$

$$\alpha_t = \sqrt{3}a \quad \alpha_s = a, \quad \alpha_h = 1/\sqrt{3} \cdot a$$

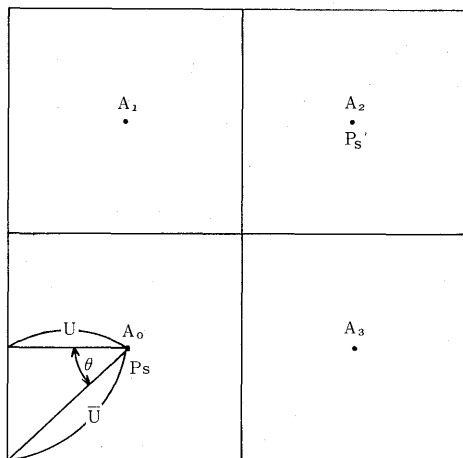
$$\beta_t = \sqrt{3}, \quad \beta_s = 1, \quad \beta_h = 1/\sqrt{3},$$

$$\gamma_t = 2.3907, \quad \gamma_s = 1.1478, \quad \gamma_h = 0.6081$$

The expression dU_i / dP_i ($i = t, s, h$) represents an estimated change in the radius of the inscribed circle of the marketing region when the firm changes its mill price. The value selected by a firm will depend on its estimate of the mill price of a rival firm. The problem of choosing which rival firm's mill price to forecast arises. In this article we will consider an estimation of the mill price (P_i' hereafter) of a rival firm located farthest among the firms surrounding the firm in question.⁷⁾ For example, in the square-shaped market area in Figure 1, the A_0 firm forecasts the mill price P_s' of the A_2 firm.

7) After this estimate is determined, an estimate about other rivals will also be uniquely determined. See also footnote 8).

Fig.1 The Forecasting Procedure of the Delivery Price of a Rival Farm



This forecasting procedure can be explained as follows. Suppose that when the firm in question lowers its mill price by one unit, the rival firm is also expected to lower the mill price by the same one unit. This situation can be described by the following.

$$\begin{aligned} dP_i'/dP_i &= 1 \\ i &= t, s, h \end{aligned} \quad (13)$$

The equations (14) and (15) below describe the situation where a rival firm shows no response and the situation where a rival firm to the contrary raises its mill price, respectively.

$$dP_i'/dP_i = 0 \quad (14)$$

$$dP_i'/dP_i = -1 \quad (15)$$

$$i = t, s, h$$

The case when $dP_i'/dP_i = 1$ corresponds to a Lösch type model, and the case when $dP_i'/dP_i = -1$ corresponds to a Greenhut-Ohta model.⁸⁾ Let us

proceed in our analysis by examining in turn the values dP_i'/dP_i as 1, -1 or $1/2$ so that our analysis will include the results derived from both the Lösch and the Greenhut=Ohta type model.

Once the values of dP_i'/dP_i ($i=t, s, h$) are determined, the values of dU_i/dP_i can be obtained in the following equation,

$$\begin{aligned} dU_i/dP_i &= (dP_i'/dP_i - 1)/2R_i \\ i &= t, s, h, \\ R_t &= 2, R_s = \sqrt{2}, R_h = 2/\sqrt{3}. \end{aligned} \tag{16}$$

In the case when $dP_i'/dP_i = 1$, namely, when it corresponds to a Lösch type model, then $dU_i/dP_i = 0$ holds. In the case when $dP_i'/dP_i = -1$, namely, when it corresponds to a Greenhut=Ohta model, $dU_t/dP_t = -1/2$, $dU_s/dP_s = -1/\sqrt{2}$, and $dU_h/dP_h = -\sqrt{3}/2$ hold. Moreover, when $dP_i'/dP_i = 1/2$, $dU_t/dP_t = -1/8$, $dU_s/dP_s = -1/4\sqrt{2}$, and $dU_h/dP_h = -\sqrt{3}/8$ hold. Substituting these values for dU_i/dP_i , according to the previous estimates of the change in a rival's mill price, will complete the equation (12), which represents the condition (B).

Since the conditions (A) and (B) above must both be satisfied in spatial competitive equilibrium, the equilibrium mill price P_i^* ($i = t, s, h$) and the radius U_i^* of an inscribed circle of the market area can be obtained

8) See Cappozza and Van Order [4]. If we consider an estimate of the mill price of a rival firm located nearest among the firms surrounding the firm in question, we get the results different from those shown by Greenhut=Ohta. That is, $dP_i'/dP_i = -1$ does not directly correspond to Greenhut=Ohta model.

by simultaneously solving the equations (11) and (12)⁹⁾. Table 4 shows each equilibrium value for the cases when $dP_1'/dP_1 = 1, 1/2$, or -1 . In Table 4, without loss of generality, the marginal cost K is, for simplicity, assumed to be zero. Moreover, the equilibrium values in each level of fixed costs were derived by increasing the fixed costs by increments of $0.01a^4$ from $0.01a^4$ to $0.1a^4$.

Table 4. The Equilibrium Values under Spatial Competitive Equilibrium

Fixed cost	Model, dP'/dP	Triangle		Square		Hexagon	
		Pt^*	Ut^*	Ps^*	Us^*	Ph^*	Uh^*
$0.01a^4$	Lösch 1	0.4557a	0.0963a	0.4583a	0.1091a	0.4589a	0.1171a
	H=S (1/2)	0.2673a	0.1065a	0.2418a	0.1249a	0.2267a	0.1371a
	G=O -1	0.1287a	0.1421a	0.1128a	0.1712a	0.1042a	0.1907a
$0.02a^4$	Lösch 1	0.4343a	0.1429a	0.4383a	0.1613a	0.4392a	0.1730a
	H=S (1/2)	0.3069a	0.1504a	0.2842a	0.1737a	0.2697a	0.1893a
	G=O -1	0.1633a	0.1885a	0.1440a	0.2254a	0.1336a	0.2502a
$0.03a^4$	Lösch 1	0.4160a	0.1827a	0.4214a	0.2055a	0.4227a	0.2202a
	H=S (1/2)	0.3239a	0.1883a	0.3054a	0.2153a	0.2927a	0.2333a
	G=O -1	0.1877a	0.2255a	0.1664a	0.2678a	0.1549a	0.2963a
$0.04a^4$	Lösch 1	0.3988a	0.2200a	0.4057a	0.2465a	0.4047a	0.2638a
	H=S (1/2)	0.3305a	0.2243a	0.3165a	0.2542a	0.3059a	0.2744a
	G=O -1	0.2068a	0.2590a	0.1844a	0.3052a	0.1723a	0.3367a
$0.05a^4$	Lösch 1	0.3818a	0.2569a	0.3905a	0.2864a	0.3925a	0.3061a
	H=S (1/2)	0.3310a	0.2601a	0.3215a	0.2924a	0.3129a	0.3145a
	G=O -1	0.2227a	0.2913a	0.1998a	0.3405a	0.1874a	0.3744a
$0.06a^4$	Lösch 1	0.3643a	0.2950a	0.3751a	0.3265a	0.3776a	0.3485a
	H=S (1/2)	0.3271a	0.2973a	0.3221a	0.3312a	0.3156a	0.3552a
	G=O -1	0.2356a	0.3242a	0.2133a	0.3752a	0.2009a	0.4110a
$0.07a^4$	Lösch 1	-----	-----	0.3590a	0.3685a	0.3621a	0.3925a
	H=S (1/2)	0.3193a	0.3377a	0.3192a	0.3720a	0.3147a	0.3976a
	G=O -1	0.2463a	0.3596a	0.2253a	0.4108a	0.2133a	0.4482a
$0.08a^4$	Lösch 1	-----	-----	0.3416a	0.4140a	0.3455a	0.4398a
	H=S (1/2)	-----	-----	0.3129a	0.4166a	0.3106a	0.4435a
	G=O -1	-----	-----	0.2358a	0.4491a	0.2244a	0.4874a
$0.09a^4$	Lösch 1	-----	-----	0.3215a	0.4666a	0.3267a	0.4933a
	H=S (1/2)	-----	-----	0.3026a	0.4683a	0.3028a	0.4958a
	G=O -1	-----	-----	0.2450a	0.4932a	0.2349a	0.5313a
$0.1a^4$	Lösch 1	-----	-----	-----	-----	0.3031a	0.5605a
	H=S (1/2)	-----	-----	-----	-----	0.2896a	0.5615a
	G=O -1	-----	-----	-----	-----	0.2444a	0.5861a

9) For a method of solving these simultaneous equations, refer to K. Suzuki, M. Kaneda and T. Ishikawa [17].

B. The Shapes of Market Areas in Spatial Competitive Equilibrium

The size of market regions in competitive equilibrium can be easily calculated from the radius of an inscribed circle of the market area listed in Table 4. Table 5 shows, for each shape, the size of market region in the levels of each fixed cost, corresponding to the three values of dP_i'/dP_i . From Table 5 the following are demonstrated. In the case of Lösch model ($dP_i'/dP_i=1$),

Table 5. Areas in the Equilibria for the Three Kind of Shapes

Fixed cost	Model, dP'/dP	Triangle	Square	Hexagon
$0.01a^4$	Lösch 1	$0.0482a^2$	$0.0476a^2$	$0.0475a^2$
	H=S (1/2)	$0.0589a^2$	$0.0624a^2$	$0.0651a^2$
	G=O -1	$0.1049a^2$	$0.1172a^2$	$0.1260a^2$
$0.02a^4$	Lösch 1	$0.1061a^2$	$0.1041a^2$	$0.1037a^2$
	H=S (1/2)	$0.1175a^2$	$0.1207a^2$	$0.1241a^2$
	G=O -1	$0.1846a^2$	$0.2032a^2$	$0.2168a^2$
$0.03a^4$	Lösch 1	$0.1734a^2$	$0.1689a^2$	$0.1679a^2$
	H=S (1/2)	$0.1842a^2$	$0.1854a^2$	$0.1886a^2$
	G=O -1	$0.2643a^2$	$0.2869a^2$	$0.3042a^2$
$0.04a^4$	Lösch 1	$0.2515a^2$	$0.2430a^2$	$0.2410a^2$
	H=S (1/2)	$0.2613a^2$	$0.2585a^2$	$0.2608a^2$
	G=O -1	$0.3485a^2$	$0.3726a^2$	$0.3928a^2$
$0.05a^4$	Lösch 1	$0.3430a^2$	$0.3281a^2$	$0.3245a^2$
	H=S (1/2)	$0.3416a^2$	$0.3420a^2$	$0.3427a^2$
	G=O -1	$0.4410a^2$	$0.4638a^2$	$0.4856a^2$
$0.06a^4$	Lösch 1	$0.4521a^2$	$0.4264a^2$	$0.4208a^2$
	H=S (1/2)	$0.4594a^2$	$0.4388a^2$	$0.4371a^2$
	G=O -1	$0.5464a^2$	$0.5631a^2$	$0.5854a^2$
$0.07a^4$	Lösch 1	-----	$0.5432a^2$	$0.5337a^2$
	H=S (1/2)	$0.5926a^2$	$0.5535a^2$	$0.5477a^2$
	G=O -1	$0.6721a^2$	$0.6750a^2$	$0.6960a^2$
$0.08a^4$	Lösch 1	-----	$0.6855a^2$	$0.6700a^2$
	H=S (1/2)	-----	$0.6942a^2$	$0.6815a^2$
	G=O -1	-----	$0.8068a^2$	$0.8229a^2$
$0.09a^4$	Lösch 1	-----	$0.8709a^2$	$0.8431a^2$
	H=S (1/2)	-----	$0.8772a^2$	$0.8515a^2$
	G=O -1	-----	$0.9730a^2$	$0.9778a^2$
$0.1a^4$	Lösch 1	-----	-----	$1.0885a^2$
	H=S (1/2)	-----	-----	$1.0925a^2$
	G=O -1	-----	-----	$1.1901a^2$

irrespective of the level of fixed costs, the area of a hexagonal market region always becomes the smallest. On the other hand in the case of Hotelling=Smithies (which is limited to $dP_i'/dP_i = 1/2$) and of Greenhut=Ohta model ($dP_i'/dP_i = -1$), as the level of fixed costs increases, the shape that makes the area of market region the smallest shifts from a triangle to a square, and further to a hexagon. A comparison of the size of each market area shape will bring forth the following conclusion: If we predict the shape of market area based on the assumption that the shape which maximizes the number of firms per unit of area materializes in a competitive equilibrium, then a triangle, square, or hexagon is possible. Furthermore, the shape that will materialize depends on the type of spatial competition and the level of fixed costs.

C. Market Area Shapes and Social Surplus

Substituting the values of the equilibrium mill prices and the radii of inscribed circles in market areas (P_i^* and U_i^* shown in Table 4) for P_i and U_i in the equation (9') will derive the value of the consumer surplus generated in each marketing region shape. Table 6 lists the amounts of the consumer surplus generated in each market area shape. The following can be demonstrated from Table 6. In the case of Lösch model ($dP_i'/dP_i = 1$), as the level of fixed costs increases, the shape that maximizes the consumer surplus per market area will shift from a triangle to a square, and further to a hexagon. In the cases of Hotelling=Smithies ($dP_i'/dP_i = 1/2$) and Greenhut=Ohta model, the shape that maximizes the consumer surplus is always a hexagon.

Table 6. The Consumer Surplus under a Spatial Competitive Equilibrium

Fixed cost	Model, dP'/dP	Triangle	Square	Hexagon
0.01a ⁴	Lösch 1	0.00504a ⁴	0.0050a ⁴	0.00502a ⁴
	H=S (1/2)	0.0119a ⁴	0.0137a ⁴	0.0150a ⁴
	G=O -1	0.0289a ⁴	0.0337a ⁴	0.0367a ⁴
0.02a ⁴	Lösch 1	0.0102a ⁴	0.0101a ⁴	0.0100a ⁴
	H=S (1/2)	0.0183a ⁴	0.0207a ⁴	0.0223a ⁴
	G=O -1	0.0411a ⁴	0.0479a ⁴	0.0522a ⁴
0.03a ⁴	Lösch 1	0.0155a ⁴	0.0153a ⁴	0.0152a ⁴
	H=S (1/2)	0.0238a ⁴	0.0264a ⁴	0.0282a ⁴
	G=O -1	0.0494a ⁴	0.0575a ⁴	0.0626a ⁴
0.04a ⁴	Lösch 1	0.0209a ⁴	0.0206a ⁴	0.0205a ⁴
	H=S (1/2)	0.0290a ⁴	0.0316a ⁴	0.0334a ⁴
	G=O -1	0.0555a ⁴	0.0645a ⁴	0.0701a ⁴
0.05a ⁴	Lösch 1	0.0267a ⁴	0.0261a ⁴	0.0259a ⁴
	H=S (1/2)	0.0343a ⁴	0.0366a ⁴	0.0383a ⁴
	G=O -1	0.0600a ⁴	0.0697a ⁴	0.0755a ⁴
0.06a ⁴	Lösch 1	0.0330a ⁴	0.0318a ⁴	0.0316a ⁴
	H=S (1/2)	0.0397a ⁴	0.0415a ⁴	0.0431a ⁴
	G=O -1	0.0637a ⁴	0.0735a ⁴	0.0794a ⁴
0.07a ⁴	Lösch 1	-----	0.0380a ⁴	0.0376a ⁴
	H=S (1/2)	0.0457a ⁴	0.0465a ⁴	0.0479a ⁴
	G=O -1	0.0667a ⁴	0.0762a ⁴	0.0819a ⁴
0.08a ⁴	Lösch 1	-----	0.0448a ⁴	0.0441a ⁴
	H=S (1/2)	-----	0.0520a ⁴	0.0529a ⁴
	G=O -1	-----	0.0779a ⁴	0.0834a ⁴
0.09a ⁴	Lösch 1	-----	0.0527a ⁴	0.0514a ⁴
	H=S (1/2)	-----	0.0582a ⁴	0.0585a ⁴
	G=O -1	-----	0.0790a ⁴	0.0838a ⁴
0.1a ⁴	Lösch 1	-----	-----	0.0608a ⁴
	H=S (1/2)	-----	-----	0.0654a ⁴
	G=O -1	-----	-----	0.0833a ⁴

Now, let us derive the social surplus per unit of area, corresponding to each shape of market area. First, the value of the consumer surplus per unit of area can be easily computed with the equation (10) and the values in Table 6.

On the other hand, since the producer surplus equates with the value of the revenue when the marginal cost is zero, deviding the revenue (which is fixed cost) by the area of market region in question shown in

Table 5 will give us the values of producer surplus per unit of area.

The social surplus per unit of area can be obtained by adding up each surplus mentioned above. The values in Table 7 show the amounts of the social surpluses per unit of area at each level of fixed costs.

Table 7. Social Welfare per unit area under Spatial Competitive Equilibrium

Fixed cost	Model, dP'/dP	Triangle	Square	Hexagon
$0.01a^4$	Lösch 1	$0.3120a^2$	$0.3155a^2$	$0.3162a^2$
	H=S (1/2)	$0.3718a^2$	$0.3804a^2$	$0.3840a^2$
	G=O -1	$0.3708a^2$	$0.3729a^2$	$0.3706a^2$
$0.02a^4$	Lösch 1	$0.2846a^2$	$0.2893a^2$	$0.2893a^2$
	H=S (1/2)	$0.3260a^2$	$0.3368a^2$	$0.3409a^2$
	G=O -1	$0.33098a^2$	$0.3341a^2$	$0.3330a^2$
$0.03a^4$	Lösch 1	$0.2624a^2$	$0.2681a^2$	$0.2692a^2$
	H=S (1/2)	$0.2921a^2$	$0.3041a^2$	$0.3086a^2$
	G=O -1	$0.3004a^2$	$0.3051a^2$	$0.3044a^2$
$0.04a^4$	Lösch 1	$0.2421a^2$	$0.2493a^2$	$0.2510a^2$
	H=S (1/2)	$0.2641a^2$	$0.2769a^2$	$0.2814a^2$
	G=O -1	$0.2740a^2$	$0.2805a^2$	$0.2803a^2$
$0.05a^4$	Lösch 1	$0.2236a^2$	$0.2319a^2$	$0.2339a^2$
	H=S (1/2)	$0.2468a^2$	$0.2531a^2$	$0.2577a^2$
	G=O -1	$0.2494a^2$	$0.2581a^2$	$0.2584a^2$
$0.06a^4$	Lösch 1	$0.2059a^2$	$0.2154a^2$	$0.2177a^2$
	H=S (1/2)	$0.2170a^2$	$0.2313a^2$	$0.2359a^2$
	G=O -1	$0.2264a^2$	$0.2371a^2$	$0.2381a^2$
$0.07a^4$	Lösch 1	-----	$0.1988a^2$	$0.2016a^2$
	H=S (1/2)	$0.1952a^2$	$0.2105a^2$	$0.2153a^2$
	G=O -1	$0.2034a^2$	$0.2165a^2$	$0.2182a^2$
$0.08a^4$	Lösch 1	-----	$0.182a^2$	$0.1852a^2$
	H=S (1/2)	-----	$0.1901a^2$	$0.1950a^2$
	G=O -1	-----	$0.1958a^2$	$0.1986a^2$
$0.09a^4$	Lösch 1	-----	$0.1639a^2$	$0.1677a^2$
	H=S (1/2)	-----	$0.1690a^2$	$0.1743a^2$
	G=O -1	-----	$0.1737a^2$	$0.1777a^2$
$0.1a^4$	Lösch 1	-----	-----	$0.1477a^2$
	H=S (1/2)	-----	-----	$0.1514a^2$
	G=O -1	-----	-----	$0.1540a^2$

Table 7 demonstrates the following: (1) The shape of a market area which

maximizes the social surplus per unit of area depends upon the types of models and the levels of fixed costs. In the Greenhut=Ohta type model, when the fixed cost is below $0.04a^4$, the social surplus per unit of area becomes the largest in a square market region. It becomes the largest always in a hexagonal market area either in a Lösch type model or in a Hotelling=Smithies type model (although it is limited to the case when $dP_i'/dP_i=1/2$). (2) In general, the social surplus per unit of area will become the largest in a Greenhut=Ohta type model. However, if the level of fixed cost is low, it becomes the largest in a Hotelling=Smithies type model ($dP_i'/dP_i=1/2$). It remains the smallest always in a Lösch type model.

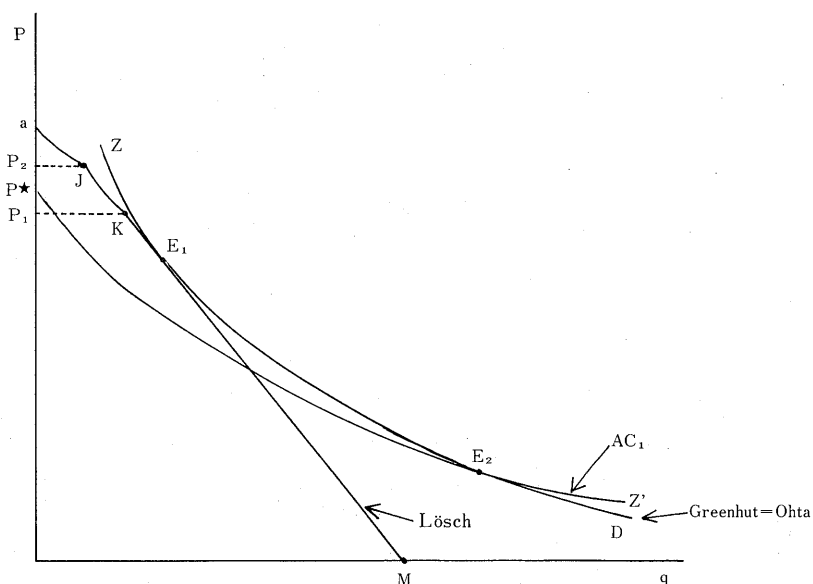
D. The Shape of Demand Curves and the Differences in Equilibrium Values

As is made clear in the preceding analysis, differences in types of models, and therefore differences in the forecasted values of the mill price of a rival firm, dP_i'/dP_i ($i = t, s, h$), will cause a substantial difference in the values of the area of the marketing region and the amounts of social surplus, even in the same shape. Another reason for this is that the values of the mill price under competitive equilibrium and the radii of an inscribed circle in the marketing region will vary according to the models. Thus, a question arises as to why in the Lösch model, that is, under the assumption that a rival firm will make the same change in its mill price as the firm in question, the equilibrium mill price becomes higher and the length of the inscribed circle to market area becomes shorter. This problem can be easily solved by analyzing the shapes of demand curves in each model (at each value of (dP_i'/dP_i)).

Now, let us take up Lösch and Greenhut=Ohta model, and analyze the demand curves for hexagonal market areas. The shapes of demand curves

in Lösch type, a Greenhut=Ohta type, and a Hotelling=Smithies type model have already been studied by Greenhut, Hwang and Ohta [9] and Batten [1] in a generalized manner. Therefore, our analysis will proceed with the assumption that each consumer has a linear demand curve as described in the equation (1). In the Lösch type model which is described by $dP_h'/dP_h = 1$ and $dU_h/dP_h = 0$ the size of market area of a firm is regarded as being independent of the mill price. The demand curve for this case takes the shape of a-J-K-M in Figure 2. An explanation is as follows: as the firm lowers its mill price from a demand price (a) to P_2 , the region containing the consumers who purchase goods can be supposed to expand within a given market area in a concentric circle pattern. Hence, the quantity demanded will, as shown in the section from point a to point J, increase in the same manner when the firm is a monopoly. Next, when the mill price

Fig. 2 The Shapes and Demand Curves for Greenhut=Ohta and Lösch Model



of goods will be suppressed by six sides of a market area, and the degree of increase in the quantity demanded will thus diminish, as indicated by the section from point J to point K. Moreover, if the mill price drops below P_1 , the purchasers of goods will be confined to given marketing region; only the consumers within this market area will increase their demand as a result of the lower mill price. Therefore, the demand curve becomes linear, starting at point K, (which corresponds to point P_1), and the increase in the quantity demanded will be minimized.

Further, let us consider the case of Greenhut=Ohta model ($dP_h'/dP_h = -1$). In this case, unlike in Lösch type model, the market area is not fixed. If the mill price goes down by one unit, the radius of the market region will expand $\sqrt{3/2}$. For this case the shape of a demand curve will be depicted as P*-D in Figure 2. The quantity demanded will increase due to two effects: an increase in the number of new purchasers and an increase in quantity demanded by the existing purchasers. Therefore, the shape of a demand curve becomes smoother and flatter than in the case of Lösch model. Thus, this demand curve becomes similar to the demand curve for the monopolistic firm which has a hexagonal market. In sum, the results of a comparison of the shapes of demand curves in two cases of Lösch and Greenhut=Ohta model ($dP_h'/dP_h = -1$) is more elastic than that for the case of Lösch model ($dP_h'/dP_h = 1$).

Having established the elasticity of different demand curves for Lösch and Greenhut=Ohta models, we can now examine the resulting equilibrium points. A competitive equilibrium occurs at the point where the average cost curve and the average revenue curve meet and where the slopes of the two tangents to these two curves are equal (this is a so-called Chamberlian solution¹⁰⁾). Suppose that the average cost curve corresponding to a produc-

tion cost can be derived (AC_1 in Figure 2). In such a case, the competitive equilibria for the cases of Lösch and Greenhut=Ohta model occur at E_1 and E_2 , respectively (as depicted in Figure 2). Since the demand curve under Lösch model is more inelastic than under Greenhut=Ohta model, the competitive equilibrium under Lösch model always takes place at the upper left-hand point on the average cost curve as opposed to the situation of Greenhut=Ohta model. Thus, in Lösch model, the equilibrium mill price is always higher, and the radius of an inscribed circle to the market area is always shorter than with the case of Greenhut=Ohta model.¹¹⁾

Next, we attempt to solve the following problem, using the preceding analysis of the shapes of demand curve. The problem is; as shown in Table 4, in the case of Lösch model, a triangular market area does not materialize when the level of fixed cost is $0.074a^4$. What is the economic reason for this?

If the marginal cost is assumed to be zero, the level of a maximum fixed cost in which a triangular market area materializes is calculated to be $0.074a^4$.¹²⁾ This value is, in itself, independent of the model. However, strictly according to the logic of a Chamberlian solution, the levels of fixed costs that a triangular market region can bear in calculation will vary as

10) Chamberlin, E. H. [5]

11) When the value for dP_i'/dP_i falls in between -1 and 1 , a competitive equilibrium will take place in between the point E_1 and E_2 on the average cost curve in Figure 2. Thus, the equilibrium mill price for this case will fall between those values of the cases when $dP_i'/dP_i = -1$ and when $dP_i'/dP_i = 1$.

12) This value of $0.074a^4$ represents, as shown in Table 1, a maximum revenue that a monopolistic firm can obtain in a triangular market area.

the values with models change.

Suppose now that AC_2 in Figure 3 represents the average production cost curve corresponding to the fixed cost of $0.074a^4$. Suppose further that an optimal triangular market area which can only generate the equal amount of a revenue ($0.074a^4$) is given to the firm. In this case, if under Lösch model the demand curve in this triangular market area will become a-J-K-M in Figure 3. Since it touches the AC_2 curve at the point K, the firm can operate in a triangular market area. But, at the point K, the slopes of the tangents to the demand curve and to the average cost curve do not coincide, and the point K is not a strict Chamberlian solution. In order for a strict Chamberlian solution to materialize the level of fixed cost has to be below $0.074a^4$, and the average cost curve has to touch the linear segment of the demand curve. In sum, following the tenet of strict

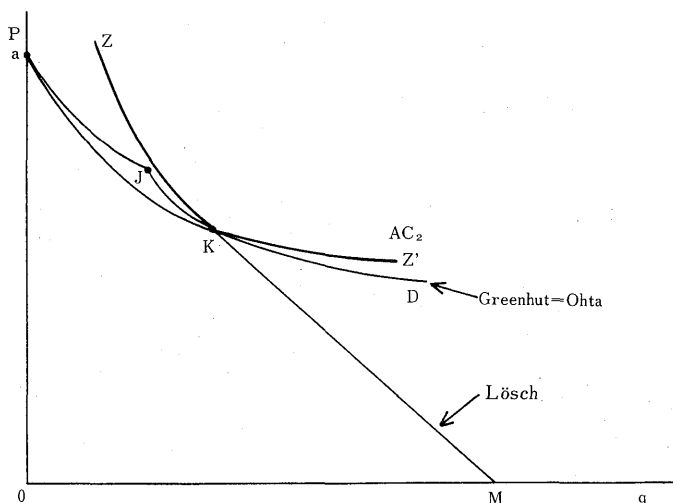


Fig.3 The Average Production Cost Curve and the Demand Curve for Greenhut=Ohta and Lösch Model.

Chamberlian solution, under Lösch model a triangular market area will not be able to bear in calculation the fixed cost of $0.074a^4$.¹³⁾

On the other hand, when $dP_t'/dP_t = -1$, the demand curve becomes a smooth-shaped curve (a-K-D) and coincides with the monopolistic firm's demand curve. It touches the AC_2 curve at point K, and the slopes of the tangents to these two curves become equal. Therefore, at the point K, the Chamberlian solution is established, and a triangular market area can bear the fixed cost of $0.074a^4$. Table 4 lists the results of computations strictly along the Chamberlian solution. The level of maximum fixed cost each triangular market area can bear in calculation will differ according to models.¹⁴⁾

13) When a market area is a triangle, under Lösch model an exact Chamberlian solution will not arise even though the fixed cost gets lowered to $0.07a^4$. More exactly, the fixed cost a triangle can bear in calculation is $0.0673a^4$. Notice this gap of $0.0673a^4$ and $0.074a^4$.

14) Our analysis in the latter half of this section applies to other shape of market area as well as to a triangular market area. That is, under Lösch model, a maximum fixed cost bearable, even when the shapes of market area are a square or a hexagon, is smaller than under the Lösch model. An economic reason for this result will be made clear by our analysis in this section.

V. CONCLUDING REMARKS

Our analysis produced the following conclusions. First, in a monopolistic equilibrium, a hexagonal market area maximizes profits, and a triangular market area makes the social surplus per unit area largest. Secondly, the shapes of market areas that materialize in competitive equilibrium can be a square, a triangular, or hexagon, if the shape is based only on the criterion of maximizing the number of firms per square mile. With regard to social surplus, the shape of market area that maximizes the social surplus per unit area is a hexagon or square. Moreover, the social surplus per unit of area will decline as the level of fixed costs increases.

If for some reason firms are allowed to limit the entry of rival firms, and if a certain form of monopoly as was explained in section III is guaranteed, triangular market area is preferred from the social viewpoint, as long as there is no constraint on the fixed cost. However, if free entry is allowed, a hexagon or square is preferable as the shape of market area for firms.

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