

PROFIT MAXIMIZATION AND FIRM'S LOCATION

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I Introduction

In an analysis of location of firms, the assumption about market condition of a product almost dictates an analytical framework. This assumption about market condition is usually divided into two categories. First category is an assumption about competition. In this case, from a view point of microeconomics, any one of perfect competition, monopolistic competition, oligopoly or monopoly is assumed. Second category is about spatial extension of market. Market is assumed as a point, a line, or a plane. Usually, under these two sets of assumptions about market various aspects of firm's location are considered.

We have, in our previous researches, assumed a perfectly competitive point market, and analyzed determinations of a minimum-transportation-cost location and of a minimum-production-cost location.^{1).2)} Relationship between firm's location and cost has been a focal point in our previous analyses. In this process we have left out consideration about relationship between firm's location, product sales and revenue. Therefore, in this paper, we will add relationship between firm's location, sales and revenue into our previous framework of analysis, thereby enabling us to consider relations of a firm's

location and profit. That is to say, we will assume a monopoly, divide market into cases of a line and a point, and will analyze which location would provide a monopoly firm with a maximum profit in each of these cases.

In Section II a linear market is assumed. And, firstly, sales area, volume of sales, and optimal prices will be considered. Secondly, under an assumption that materials-producing sites exist in both sides of a linear market, we will analyze how sales, revenue, cost and profit would change as a location of a firm shifts, and will derive a location which maximizes profit. In Section III, a point market and two materials-producing sites are assumed in a plane, and we will determine a combination of location and price which maximizes a monopoly firm's profit. Section IV will summarize our analysis and conclude the paper.

II Optimal Location in Linear Space

Considerations in this section are divided into the following two cases. First, we will neglect a firm's cost, will focus on a firm's revenue, and will analyze a firm's sales, marketing areas and prices. Secondly, an aspect of cost will be added to the analysis above, and relationship between a firm's location and profit will be examined.

1 Assumptions

Basic assumptions in this section are stated as follows.

- 1) A market is linear.
- 2) In this linear market consumers are evenly distributed. Each consumer owns an identical demand function in the following.

$$q = a - b(p + tu) \quad (1)$$

where q is amount of purchase, p is a firm's mill price, u is a distance from a consumer to a firm, t is a freight rate, and a and b are positive parameters. In

the following it is assumed for simplicity that both b and t are 1, and that the value of a is sufficiently large.

3) Transportation cost is borne by a consumer.

4) In a linear market one firm exists and it behaves according to a principle of profit maximization.

In the following, assumptions required will be shown at the beginning of each section.

2 (a) Firm's Location in an Infinitely Linear Market

Here, let us assume that market is linear and that a firm incurs no cost. Now let us suppose that a firm is located at a given point S in an infinite line, as shown in Figure 1 (A), and is engaged in a sale of its product. A consumer is assumed, following the equation (1), to own a demand function, a line $a-q$ as depicted in Figure 1 (B). In this case, a firm's marketing area and total sales will be determined as follows.

When a firm sells its product at a mill price p_1 , a delivery price, $p_1 + u$, will go up according to a distance from a firm, as lines p_1-m or p_1-n in Figure 1 (A). And, the quantity demanded becomes zero at M and N , the points at which a delivery price is just equated with a consumer's demand price a . The points M and N are the end points of marketing area. Therefore, when the mill price is p_1 , a firm's marketing area is an area $M-N$. The distance from a firm's location S to the end points of marketing area M or N is called a radius of marketing area, and is shown as U . Marketing area when a mill price increases to p_2 is determined in a similar manner, and is shrunk to an area $M'-N'$. Next, using the equation (1), a length of a monopoly firm's marketing area can be described as follows. That is, since the quantity demanded becomes exactly zero at the end points of marketing area, the following equation is established at those end points.

$$0 = a - (p + u) \quad (2)$$

Therefore, a radius U of marketing area is expressed as

$$U = a - p \quad (3)$$

Thus, marketing area is expressed as $2(a - p)$.

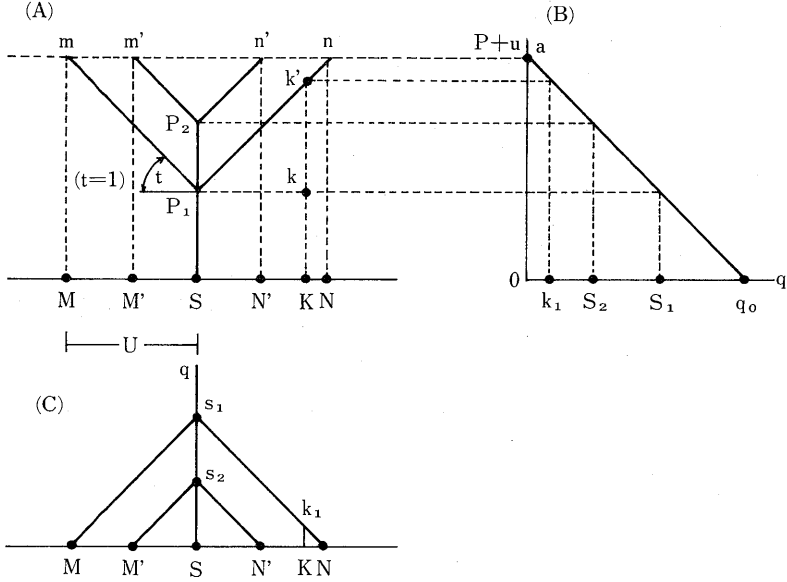


Figure 1 Formation of Marketing Area in Infinite Linear Market

Now, if a firm sets up a mill price at P_1 and sells its good, its delivery price goes up as $p_1 + n$. Therefore, the amount of purchase by a consumer will decrease as a distance from a firm gets far, along a demand curve shown in Figure 1 (B). Since a consumer situated next to a point S can purchase the good at a mill price, he will buy a maximum amount, depicted as S_1 in Figure 1 (B). A consumer at point K pays a transportation cost of $k' - k$. Thus, its delivery price becomes k' and his amount of purchase is k_1 in Figure 1 (B). Amounts of purchases by each consumer are shown in Figure 1 (C). Total sales in the case when a firm sets its mill price at p_1 is shown as an area of a

triangle MS_1N in Figure 1 (C). Similarly, when a mill price is p_2 , a firm's total sales amount to a triangle $M'S_2N'$.

A firm's total amount of sales Q can be expressed, using the equation (1), in the following equation (4).

$$Q = 2 \int_0^U [a - (p + u)] du \quad (4)$$

Since U is expressed as $a - p$, total sales Q can be written as follows as a function of mill price p .

$$Q = (a - p)^2 \quad (5)$$

Next, let us consider a firm's optimal mill price and optimal marketing area. Here, cost is assumed to be zero, and a mill price which brings a maximum revenue (R) will be derived. Revenue (R) can be expressed as the following equation (6).

$$R = P \cdot Q \quad (6)$$

An optimal mill price can be obtained by differentiating the equation (6) with respect to p and by equating it to zero. Namely,

$$dR/dP = a^2 - 4ap + 3p^2 = 0 \quad (7)$$

Thus, an optimal mill price p^* is

$$p^* = a/3 \quad (8)$$

Moreover, an optimal marketing area becomes, using $2(a - p)$, $4a/3$. Total sales in this case is $(2a/3)^2$. Furthermore, in this case, a location of a firm is of course at the center of marketing area.

2 (b) A Firm's Location in a Finite Linear Market

In this section we will consider a firm's location and an optimal price in a finite linear market, a length, as shown in Figure 2 (A). Here it is also assumed that a firm's cost is zero.

Now let us suppose that a firm is situated at a point S , a given distance x

from a point M in Figure 2 (A) and that it sells its good at a mill price p_1 . Amounts of purchases by each consumer is determined by a demand curve, as shown in Figure 2 (B). Thus a firm's total sales Q are described as in the following equation (9).

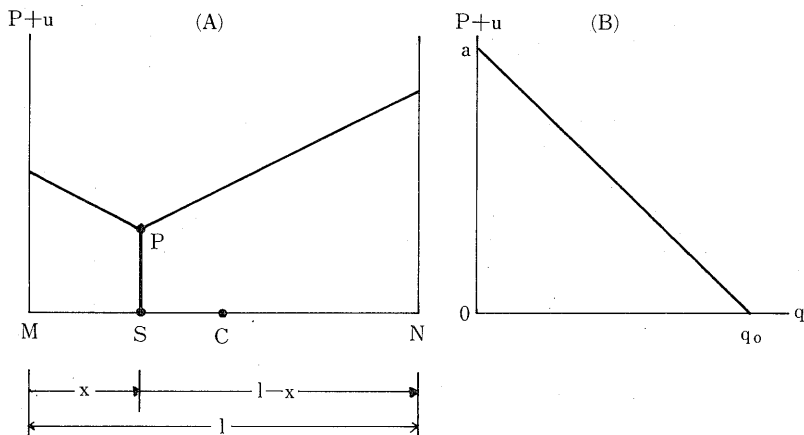


Figure 2 Firm's Location in Finite Linear Market

$$Q = \int_0^x [a - (p+u)] du + \int_0^{l-x} [a - (p+u)] du \quad (9)$$

A firm's revenue (R) is obtained by multiplying p with the equation (9) as follows.

$$R = p \cdot Q \quad (10)$$

An optimal location of a firm is obtained by differentiating the equation (10) with respect to x and by equating it to zero. Namely,

$$dR/dx = -2px + lp = 0 \quad (11)$$

Therefore an optimal location x^* is derived as follows.

$$x^* = l/2 \quad (12)$$

This tells that a firm's optimal location lies at the center of a given market. This means that in Figure 2 (A) a firm will locate at a center C in the market

M-N.

Next, differentiating the equation (10) with respect to p and equating the result to zero give us a firm's optimal mill price:

$$dR/dP = -x^2 + lx - 2lp - l^2/2 + al = 0 \quad (13)$$

Therefore,

$$p = 1/2l(-x^2 + lx - l^2/2 + al) \quad (14)$$

Since an optimal location is $l/2$, substitution of $l/2$ into x gives an optimal mill price p^* as follows.

$$p^* = a/2 - l/8 \quad (15)$$

As shown in the equation (15), a firm's optimal price depends on the length of market, and as a length gets longer, it goes down. This certainly differs from a firm's pricing in an infinite linear market.

3 A Firm's Location and Profit in a Finite Linear Market

As was clear from our considerations in a previous section, a firm's location which makes revenue the largest in a linear market is a center of marketing area. On the other hand, as our previous discussion on location and cost clearly demonstrated, a firm's location which incurs a minimum cost in a linear space lies at end points of a linear space.^{3),4)} It follows that a firm's location will differ, depending on which viewpoint (cost or revenue) is taken. Therefore, in this section the following problem will be considered. Namely, if there exist material sites at A and B in Figure 3 and consumers are distributed along a line AB, which location will make a firm maximize its profit? To solve this problem some more assumptions will be added to our basic assumptions mentioned in Section 1.

3 (a) Assumptions

1) Good Z sold by a firm is produced with mutually substitutable materials A and B. Production function is written as follows.

$$Q_Z = \gamma A^\alpha B^\beta \quad (16)$$

where Q_Z is amount of Z produced, γ , α , and β are positive constants, with a relation that $\alpha + \beta = 1$. Therefore, a production function is of homogeneous degree 1. In the following we assume that equals 1.

2) Materials A and B are produced at A and B, respectively, in Figure 1. Mill prices of each material are P_A and P_B respectively.

3) Freight rates of materials A and B are 1, and transportation costs are borne by a firm.

4) Consumers are evenly distributed on a line connecting material sites A and B. The length of a line is 1.

Under these assumptions above and previous assumptions in Section 1 we will analyze relationship between a firm's location and profit. In the following, we will also take a close look at what impact prices of materials A and B would give the relationship between location and profit.

3 (b) A Firm's Location and Profit

First, let us clarify relationship between a firm's location and profit. Now, in Figure 3, if a firm's location S is expressed as a distance x from a material site A, a firm's revenue R is, by multiplying a mill price (this is simply called "price" hereafter) with sales calculated in the equation (9), expressed in the following.

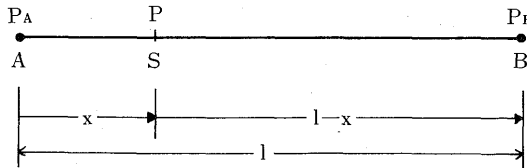


Figure 3 Finite Linear Market and Material Producing Site

$$R = p \left[\int_0^x (a - (p+u)) du + \int_0^{1-x} (a - (p+u)) du \right] \quad (17)$$

$$R = p(-x^2 + lx - l^2/2 - lp + al) \quad (18)$$

Next, cost will be calculated as follows. First, for the use of optimal materials A and B, since price ratios of each material is equated with ratios of marginal productivities of each material, the following equation (19) will be established for a firm.

$$\frac{x + P_A}{(1-x) + P_B} = \frac{\alpha A^{\alpha-1} B^\beta}{\beta A^\alpha B^{\beta-1}} \quad (19)$$

The equation (19) can be transformed as follows.

$$\frac{x + p_A}{(1-x) + p_B} = \frac{\alpha}{\beta} \left(\frac{Q_Z}{A} \right)^{1/\beta} \quad (20)$$

$$\frac{x + p_A}{(1-x) + p_B} = \frac{\alpha}{\beta} \left(\frac{B}{Q_Z} \right)^{1/\alpha} \quad (21)$$

From the equations (20) and (21), quantities used of materials A and B, A^* , B^* , are shown, respectively, in the following.

$$A^* = \left[\frac{(1-x) + p_B}{x + p_A} \cdot \frac{\alpha}{\beta} \cdot Q_Z^{1/\beta} \right]^\beta \quad (22)$$

$$B^* = \left[\frac{x + p_A}{(1-x) + p_B} \cdot \frac{\beta}{\alpha} \cdot Q_Z^{1/\alpha} \right]^\alpha \quad (23)$$

Therefore, a firm's cost C is, multiplying quantities used of each material with its delivered price, shown in the following equation (24).

$$\begin{aligned} C = & \left[\frac{(1-x) + p_B}{x + p_A} \cdot \frac{\alpha}{\beta} \cdot Q_Z^{1/\beta} \right]^\beta (p_A + x) \\ & + \left[\frac{x + p_A}{(1-x) + p_B} \cdot \frac{\beta}{\alpha} \cdot Q_Z^{1/\alpha} \right]^\alpha ((1-x) + p_B) \end{aligned} \quad (24)$$

The equation (24) can be transformed as

$$C = ((1-x) + p_B)^{1-\alpha} (x + p_A)^\alpha \left(\frac{\alpha}{\beta}\right)^{1-\alpha} Q_z \\ + ((1-x) + p_B)^{1-\alpha} (x + p_A)^\alpha \left(\frac{\alpha}{\beta}\right)^{-\alpha} Q_z \quad (25)$$

It follows that

$$C = Q_z ((1-x) + p_B)^{1-\alpha} (x + p_A)^\alpha \left[\left(\frac{\alpha}{\beta}\right)^{1-\alpha} + \left(\frac{\alpha}{\beta}\right)^{-\alpha} \right] \quad (26)$$

Since Q_z is given by the equation (9), a firm's cost C is expressed as a function of x below.

$$C = (-x^2 + lx - l^2/2 - lp + la)(1-x + p_B)^{1-\alpha} (x + p_A)^\alpha \cdot \\ \left(\left(\frac{\alpha}{\beta}\right)^{1-\alpha} + \left(\frac{\alpha}{\beta}\right)^{-\alpha} \right) \quad (27)$$

From the equations (18) and (27), a firm's profit can be written such as

$$\pi = (-x^2 + lx - l^2/2 - lp + la) \cdot \\ \left[p - (1-x + p_B)^{1-\alpha} (x + p_A)^\alpha \left(\left(\frac{\alpha}{\beta}\right)^{1-\alpha} + \left(\frac{\alpha}{\beta}\right)^{-\alpha} \right) \right] \quad (28)$$

3 (c) A Firm's Location and Optimal Price

A firm's profit is, as shown in the equation (28), expressed as a function of a firm's location x and price of its good p . Thus, let us analyze how optimal price would change as a firm's location shifts, by deriving optimal price of good from the equation (28). Optimal price p^* can be obtained by differentiating the equation (28) with respect to p and by equating it to zero.

$$d\pi/dp = (-x^2 + lx - l^2/2 - pl + al) \\ + \left[p - (1-x + p_B)^{1-\alpha} (p_A + x)^\alpha \cdot \left(\left(\frac{\alpha}{\beta}\right)^{1-\alpha} + \left(\frac{\alpha}{\beta}\right)^{-\alpha} \right) \right] (-1) \\ = 0 \quad (29)$$

Optimal price p^* can be calculated, from the equation (29), as follows.

$$p^* = 1/2l \left[(-x^2 + lx - l^2/2 + al) + l(1-x+p_B)^{1-\alpha}(x+p_A)^\alpha \cdot \left(\left(\frac{\alpha}{\beta} \right)^{1-\alpha} + \left(\frac{\alpha}{\beta} \right)^{-\alpha} \right) \right] \quad (30)$$

Of course, here we will presuppose such a value of a that makes the first term in the bracket of the equation (30). Now, this optimal price p^* is a function of a firm's location x . Differentiation of p^* with respect to x will enable us to tell how p^* would change as a firm shifts its location. First differentiation is given in the following.

$$dp^*/dx = -2x + l + l \left(\left(\frac{\alpha}{\beta} \right)^{1-\alpha} + \left(\frac{\alpha}{\beta} \right)^{-\alpha} \right) \cdot \left[(1-x+p_B)^{1-\alpha} \alpha (x+p_A)^{\alpha-1} - (1-\alpha)(1-x+p_B)^{-\alpha} (x+p_A)^\alpha \right] \quad (31)$$

Therefore, if $\alpha = \beta$ and $p_A = p_B$, a curve of p^* will have its maximum at the center of market. Next, second differentiation is given as follows.

$$d^2p^*/dx^2 = -2 - l \left(\left(\frac{\alpha}{\beta} \right)^{1-\alpha} + \left(\frac{\alpha}{\beta} \right)^{-\alpha} \right) \left[\alpha(1-\alpha) \cdot \frac{(x+p_A)^\alpha}{(1-x+p_B)^{1+\alpha}} \cdot \left\{ 1 + \frac{1-x+p_B}{x+p_A} \right\} \right]^2 \quad (32)$$

As is known from the equation (32), this second differentiation becomes negative. Thus, optimal price p^* becomes a concave function, as shown in Figure 4. It is assumed in the curves of $a-b$ and $a'-b'$ that $l = 2$, $\alpha = \beta = 0.5$, and $P_A = P_B = 1.5$, or, $\alpha = \beta = 0.5$ and $P_A = P_B = 0.15$.

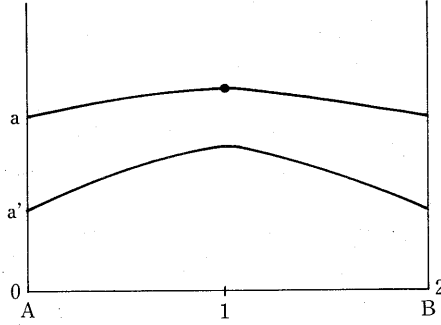


Figure 4 Firm's Location and Optimal Pricing

3 (d) Firm's Location and Amount of Sales

Quantity sold of good Q_z is shown as $-x^2 + lx - l^2/2 - lp + a$, and price which a firm imposes upon is given by the equation (30). Therefore, Q_z is obtained as a function of a firm's location x .

Namely, substitution of the equation (30) into p gives the following equation (33).

$$Q_z = 1/2 \left[-x^2 + lx - l^2/2 + a - l(1-x+p_B)^{1-\alpha} \cdot (x+p_A)^\alpha \left(\left(\frac{\alpha}{\beta} \right)^{1-\alpha} + \left(\frac{\alpha}{\beta} \right)^{-\alpha} \right) \right] \quad (33)$$

Differentiation of Q_z with respect to x gives us a clear picture of how amount of sales of good would change according to a firm's location. First differentiation is given as

$$dQ_z/dx = -x + l/2 - l/2 \left(\left(\frac{\alpha}{\beta} \right)^{1-\alpha} + \left(\frac{\alpha}{\beta} \right)^{-\alpha} \right) \cdot \left[-(1-\alpha)(1-x+p_B)^{-\alpha}(x+p_A)^\alpha + \alpha(1-x+p_B)^{1-\alpha}(x+p_A)^{\alpha-1} \right] \quad (34)$$

Furthermore, second differentiation is written as

$$d^2 Q_z / dx^2 = -1 + 1/2 \left(\left(\frac{\alpha}{\beta} \right)^{1-\alpha} + \left(\frac{\alpha}{\beta} \right)^{-\alpha} \right).$$

$$\left[\alpha(1-\alpha)(1-x+p_B)^{-(1+\alpha)}(x+p_A)^\alpha \cdot \left\{ 1 + \frac{1-x+p_B}{x+p_A} \right\}^2 \right] \quad (35)$$

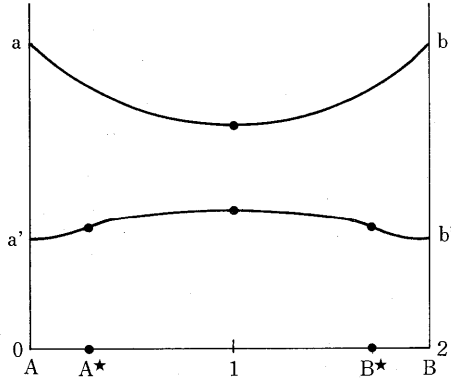


Figure 5 Firm's Location and Amount of Sales

Signs of first and second differentiations are obtained from the equations (34) and (35). These would vary, when the values of l , α and β are assumed constant, according to prices of materials A and B, P_A and P_B . Namely, sales quantity Q_z can be both convex and concave functions. Curves a-b and a'-b' in Figure 5 show typical movements of Q_z . For curves a-b and a'-b', when $l = 2$, $\alpha = \beta = 0.5$ and $a = 10$, it is respectively assumed that $P_A = P_B = 0.15$ and $P_A = P_B = 1.5$. As is clear from the equation (36), Curve a'-b' in Figure 5 will have a point of inflection at points A* and B*. Heed should also be taken to the following fact: a location that attains a maximum amount sales will vary, depending upon the values of material prices; that is, when material price is low, a maximum-sales-location will be situated at end points of a linear space.

3 (e) Firm's Location and Revenue

A firm's revenue R is written as $R = p(-x^2 + lx - l^2/2 - pl + al)$. By

substituting the equation (30) into p in this expression, revenue can be expressed as a function of a firm's location x. That is to say, revenue is given as

$$R = 1/l \left[(-x^2/2 + lx/2 - l^2/4 + al/2)^2 - \left\{ 1/2(1-x+p_B)^{1-\alpha}(p_A+x)^\alpha \left(\left(\frac{\alpha}{\beta} \right)^{1-\alpha} + \left(\frac{\alpha}{\beta} \right)^\alpha \right)^2 \right\} \right] \quad (36)$$

How this R would vary as a firm's location shifts will be known by differentiating the equation (36) with respect to x. First differentiation results in the following (37).

$$\begin{aligned} dR/dx = & (-x^2 + lx - l^2/2 + alx)(-x + l/2) \\ & + \left[1/2(1-\alpha)(1-x+p_B)^{-\alpha}(x+p_A)^\alpha \right. \\ & \left. - \alpha l/2(1-x+p_B)^{1-\alpha}(x+p_A)^{\alpha-1} \right] \left(\left(\frac{\alpha}{\beta} \right)^{1-\alpha} + \left(\frac{\alpha}{\beta} \right)^\alpha \right) \end{aligned} \quad (37)$$

Furthermore, through differentiation of the equation (37) with respect to x, second differentiation becomes

$$\begin{aligned} d^2R/dx^2 = & -(-x^2 + lx - l^2/2 + al) \\ & + 2(-x + l/2)(-x + l/2) + (1-x+p_B)^{-\alpha}(x+p_A)^{\alpha-1} \\ & \left[1 + (1-x+p_B)(x+p_A) \right] - (1-x+p_B)^{-\alpha}(x+p_A)^{\alpha-1} \\ & \left[1 + (1-x+p_B)(x+p_A) \right] \end{aligned} \quad (38)$$

The next equation is derived from this equation (38).

$$d^2R/dx^2 = x^2 - 3xl - al \quad (39)$$

Since the equation (39) is obviously negative, revenue R becomes a concave curve. The curves a-b and a'-b' in Figure 6 show, when $l = 2$, $\alpha = \beta = 0.5$, cases for $P_A = P_B = 0.15$ and $P_A = P_B = 1.5$, respectively. Characteristics of revenue R curve are as follows. Namely, in curves of optimal prices and sales, difference between the highest and the lowest values of curves are

larger in case when material prices P_A and P_B are low than in case when they are high. However, in a revenue curve, regardless of values of P_A and P_B , difference remains constant.

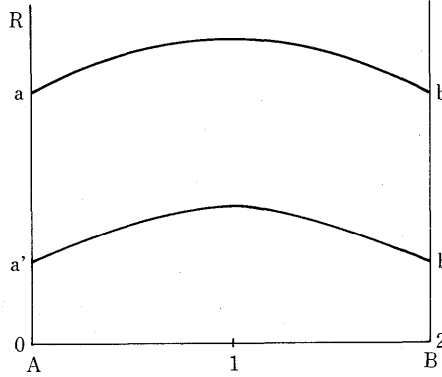


Figure 6 Firm's Location and Revenue

3 (f) Firm's Location and Cost

A firm's cost is given as $(-x^2 + lx - l^2/2 - pl + al) \cdot (1-x+p_B)^{1-\alpha}(x+p_A)^\alpha \left(\left(\frac{\alpha}{\beta} \right)^{1-\alpha} + \left(\frac{\alpha}{\beta} \right)^{-\alpha} \right)$. Substitution of the equation (30) into p provides us the following firm's cost as a function of x .

$$\begin{aligned}
 C = & (-x^2/2 + lx/2 - l^2/4 + al/2) \cdot \left[(1-x+p_B)^{1-\alpha}(x+p_A)^\alpha \cdot \right. \\
 & \left. \left(\left(\frac{\alpha}{\beta} \right)^{1-\alpha} + \left(\frac{\alpha}{\beta} \right)^{-\alpha} \right) \right] - 1/2 \left[(1-x+p_B)^{2(1-\alpha)} \cdot (x+p_A)^{2\alpha} \left(\left(\frac{\alpha}{\beta} \right)^{1-\alpha} \right. \right. \\
 & \left. \left. + \left(\frac{\alpha}{\beta} \right)^{-\alpha} \right)^2 \right] \quad (40)
 \end{aligned}$$

Movements of cost against a firm's location shifts can be obtained by differentiating the equation (40) with respect to x . First derivative becomes as follows.

$$\begin{aligned}
dC/dx &= (1-x+p_B)^{-\alpha} (x+p_A)^\alpha \left(\left(\frac{\alpha}{\beta} \right)^{1-\alpha} + \left(\frac{\alpha}{\beta} \right)^\alpha \right) \cdot \\
&\left[(1-x+p_B)(-x+1/2) - (1-\alpha)/2 (-x^2+lx-l^2/2+al) \right. \\
&\quad \left. + \alpha/2 (1-x+p_B)(x+p_A)^{-1} \cdot (-x^2+lx-l^2/2+al) \right] \\
&- 1 \left(\left(\frac{\alpha}{\beta} \right)^{1-\alpha} + \left(\frac{\alpha}{\beta} \right)^\alpha \right)^2 \cdot (1-x+p_B)^{1-2\alpha} (x+p_A)^{2\alpha} \cdot \\
&\left[\alpha(1-x+p_B)(x+p_A)^{-1} - (1-\alpha) \right] \tag{41}
\end{aligned}$$

Equating the equation (41) to zero and solving it for x provide us with a maximum of a firm's cost curve. Suppose that $\alpha = \beta$ and that $P_A = P_B$. Then the maximum is $1/2$. Differentiation of the equation (41) with respect to x gives us the following second derivative.

$$\begin{aligned}
d^2C/dx^2 &= (1-x+p_B)^{-\alpha} (x+p_A)^\alpha \left(\left(\frac{\alpha}{\beta} \right)^{1-\alpha} + \left(\frac{\alpha}{\beta} \right)^\alpha \right) \cdot \\
&\left[-2(1-\alpha) \cdot (-x+1/2) + 2(1-x+p_B)(x+p_A)^{-1} \alpha (-x+1/2) \right. \\
&\quad - (1-x+p_B) - \alpha(1-\alpha)(-x^2+lx-l^2/2+al)(x+p_A)^{-1} \\
&\quad - \frac{\alpha(1-\alpha)}{2} (-x^2+lx-l^2/2+al)(1-x+p_B)^{-1} \\
&\quad + \frac{\alpha(\alpha-1)}{2} (-x^2+lx-l^2/2+al)(1-x+p_B)(x+p_A)^{-2} \\
&\quad \left. - 1(1-x+p_B)^\alpha (x+p_A)^\alpha \left(\left(\frac{\alpha}{\beta} \right)^{1-\alpha} + \left(\frac{\alpha}{\beta} \right)^\alpha \right) (2\alpha-1) \cdot \right. \\
&\quad \left. \left\{ (1-x+p_B) \cdot (x+p_A)^{-2} + (1-\alpha) \right\} \right] \tag{42}
\end{aligned}$$

Sign of the equation (42) can not be easily determined because there are cases where several vertexes hold positive values. However, it is easily determined that the equation (42) will hold a positive value if a in the incisively negative term takes a sufficiently large value.

Therefore, it follows that a firm's cost will vary, against location shift, in a shape of a concave function. $a-b$ and $a'-b'$ in Figure 7 show, when $1 = 2$ and $\alpha = \beta = 0.5$, cost curves for cases where $p_A = p_B = 1.5$ and $p_A = p_B = 0.15$, respectively. Characteristics of this cost curve are such that difference between highest and lowest values of a curve become, as in the case of an optimal-price curve, larger when material prices are low. As is known from $a'-b'$ curve in Figure 7, when a firm locates itself at end points of a linear space, cost will become smaller than in case when a firm locates itself at a center. As for this point, cost will vary in a way different from revenue

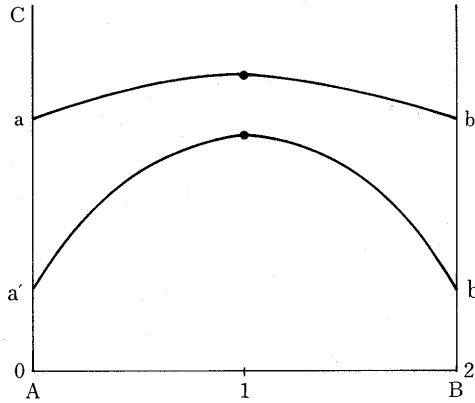


Figure 7 Firm's Location and Cost

3 (g) Firm's Location and Profit

Lastly, let us examine how a firm's profit would vary according to location shift. Profit was previously shown as $\pi = (-x^2 + lx - l^2/2 + lp + la) \cdot$

$$\left[p - (1-x+p_B)^{1-\alpha} (x+p_A)^\alpha \left(\left(\frac{\alpha}{\beta} \right)^{1-\alpha} + \left(\frac{\alpha}{\beta} \right)^{-\alpha} \right) \right].$$

Profit is expressed in the following, by substituting the equation (30) for p .

$$\pi = 1/1 \left[1/2 (-x^2 + lx - l^2/2 + al) - 1/2 \left\{ (1-x+p_B)^{1-\alpha} \cdot (x+p_A)^\alpha \cdot \left(\left(\frac{\alpha}{\beta} \right)^{1-\alpha} + \left(\frac{\alpha}{\beta} \right)^{-\alpha} \right) \right\} \right]^2 \quad (43)$$

Changes of profit when a firm shifts its location can be known by differentiating the equation (43) with respect to x . First derivative is as follows.

$$\begin{aligned} d\pi/dx = & 2/1 \left[1/2 (-x^2 + lx - l^2/2 + al) \right. \\ & \left. - 1/2 \left\{ (1-x+p_B)^{1-\alpha} (x+p_A)^\alpha \cdot \left(\left(\frac{\alpha}{\beta} \right)^{1-\alpha} + \left(\frac{\alpha}{\beta} \right)^{-\alpha} \right) \right\} \right] \\ & \left[-x + l/2 - 1/2 \left(\left(\frac{\alpha}{\beta} \right)^{1-\alpha} + \left(\frac{\alpha}{\beta} \right)^{-\alpha} \right) \cdot \right. \\ & \left. \left\{ -(1-\alpha)(1-x+p_B)^{-\alpha} (x+p_B)^\alpha + (1-x+p_B)^{1-\alpha} (x+p_A)^{\alpha-1} \right\} \right] \end{aligned} \quad (44)$$

A sign of this equation (44) depends on values of l , α , β , p_A , p_B and x . The second derivative can be easily calculated by furthermore differentiating the equation (44) with respect to x . However, we will omit it here because an equation becomes too long. A sign of $d^2\pi/dx^2$ also depends on values of l , α , β , p_A , p_B and x , and can take any value of a positive, a negative, or zero. Therefore, a firm's profit can be, as for location shift, a curve which holds a characteristic of either concavity, convexity, or both of them. $a-b$, $a'-b'$, and a^*-b^* in Figure 8 show typical curves of a firm's profit, respectively. In each curve of Figure 8, it is assumed that under $l=2$, $\alpha=\beta=0.5$, $p_A=p_B=0.15$, $\alpha=\beta=0.5$, $p_A=p_B=1.5$, and $\alpha=0.55$, $\beta=0.45$, $p_A=1.5$ and $p_B=1.1$. When $\alpha=\beta$ and $p_A=p_B$, the following can be said. When materials prices p_A and p_B are high, profit is maximized at a center of a linear space. On the other hand, when p_A and p_B are low, profit is maximized at end points of a linear space.

Next, when $\alpha \neq \beta$ and $p_A \neq p_B$, a firm's profit curve can take various forms. Therefore, profit is maximized at many points. Thus, points of profit maximization are not limited to a center or end points of a linear space, but

can be a point in between end points and a center, as shown point w of $a^* - b^*$ curve in Figure 8.⁵⁾

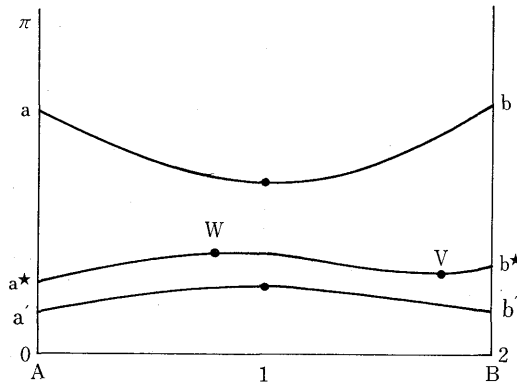


Figure 8 Firm's Location and Profit

It is clear that low and high values of materials prices p_A and p_B will have an important effect on relation between a firm's location and profit. By the way, low and high values of materials prices have a length of market as a standard. Hence, even if a material price remains constant, different lengths of market cause profit-maximizing points to vary. Curves $a-b$, $a'-b'$ and a^*-b^* show market-expansion effect. Under the condition that $\alpha = \beta = 0.5$ and $p_A = p_B = 1.5$, it is respectively assumed that $l = 2$, $l = 1$, and $l = 4$. Each curve in Figure 9 tells us the following. Length of market not only causes level of firm's profit to vary, but also causes profit-maximizing points to shift. That is to say, when the length of market is 2, profit level is the highest, and as market length becomes 4, a profit-maximizing point will shift from a center of a linear market to end points. Of course, our results here of differences in market lengths can be applied to analysis of locational effects of changes in freight rates of goods.

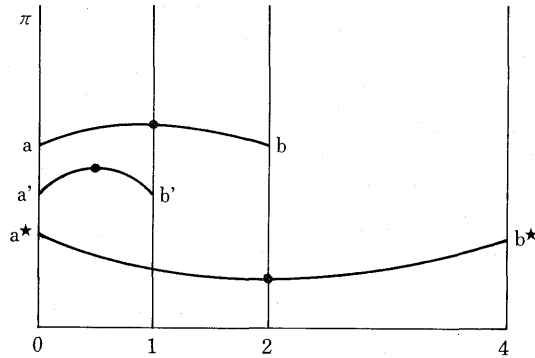


Figure 9 Firm's Location and Length of Linear Market

III Optimal Location in Plane Space

Here, we will analyze a firm's optimal location in plane space. Namely, a point which maximizes a firm's profit will be determined. Assumptions different from those presented in 1 and 3(a) of the previous section are as follows.

1 Assumptions

1) there are two materials-producing sites A and B in a plane space, and their locations are depicted at coordinates $(C, 0)$ and (W, V) (Refer to Figure 10).

2) Market is depicted at point Z, and its location is shown by the coordinates $(0, 0)$ (Refer to Figure 10).

3) Aggregate demand function in market is given as

$$Q_z = K - (p + d_z) \quad (45)$$

where K is a positive constant and d_z represents distance from a firm's location L to market.

The problem here is, under these assumptions above added, to

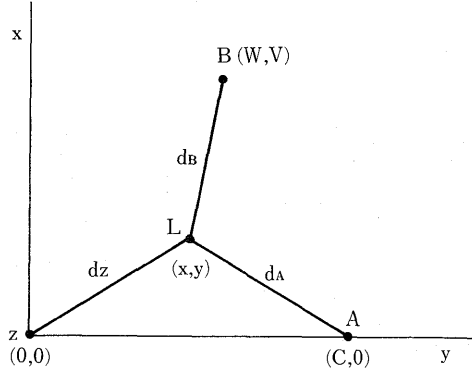


Figure 10 Firm's Location in Plane Space

determine a location $L(x^*, y^*)$ which maximizes a firm's profit.⁷⁾

2 Firm's Profit

A firm's profit is calculated by subtracting cost from revenue, and is given as

$$\pi = p(K - (p + d_z)) - [A^*(p_A + d_A) + B^*(p_B + d_B)] \quad (46)$$

where A^* and B^* are the quantities used of materials A and B, d_A and d_B are distances from materials-producing sites A and B to a firm's location. The quantities used A^* and B^* of materials A and B can be obtained by a manner similar to derivation of the equations (22) and (23), shown in the following.

$$A^* = \left[\frac{p_B + d_B}{p_A + d_A} \cdot \frac{\alpha}{\beta} \cdot (K - (p + d_z))^{1/\beta} \right]^\beta \quad (47)$$

$$B^* = \left[\frac{p_A + d_A}{p_B + d_B} \cdot \frac{\beta}{\alpha} \cdot (K - (p + d_z))^{1/\alpha} \right]^\alpha \quad (48)$$

3 Optimal Location and Determination of Optimal Price

Variables which a firm has to determine in order to maximize its profit

are a location coordinate x and y , and a product's mill price p . Hence, a firm is, in order to maximize profit, going to seek x , y , and p that simultaneously satisfy the following three equations. These x , y and p are obtained by solving the following simultaneous equations

$$\frac{\partial \pi}{\partial x} = 0 \quad (49)$$

$$\frac{\partial \pi}{\partial y} = 0 \quad (50)$$

$$\frac{\partial \pi}{\partial p} = 0 \quad (51)$$

First, as for the equation (49),

$$\begin{aligned} \frac{\partial \pi}{\partial x} = & -p \frac{\partial d_z}{\partial x} - \left[(p_A + d_A) \frac{\partial A^*}{\partial x} + (p_B + d_B) \frac{\partial B^*}{\partial x} \right] \\ & - \left[A^* \frac{\partial d_A}{\partial x} + B^* \frac{\partial d_B}{\partial x} \right] = C \end{aligned} \quad (52)$$

This equation (52) becomes the following.

$$\begin{aligned} \frac{\partial \pi}{\partial x} = & -p \frac{\partial d_z}{\partial x} - \left[(p_A + d_A) \frac{\partial A^*}{\partial (p_A + d_A)} \frac{\partial (p_A + d_A)}{\partial x} \right. \\ & \left. + (p_B + d_B) \frac{\partial B^*}{\partial (p_B + d_B)} \cdot \frac{\partial (p_B + d_B)}{\partial x} \right] \\ & - \left[(p_A + d_A) \frac{\partial A^*}{\partial (p_B + d_B)} \cdot \frac{\partial (p_B + d_B)}{\partial x} + (p_B + d_B) \frac{\partial B^*}{\partial (p_B + d_B)} \cdot \right. \\ & \left. \frac{\partial (p_B + d_B)}{\partial x} \right] \\ & - \left[(p_A + d_A) \frac{\partial A^*}{\partial Q_z} \cdot \frac{\partial Q_z}{\partial x} + (p_B + d_B) \frac{\partial B^*}{\partial Q_z} \cdot \frac{\partial Q_z}{\partial x} \right] \\ & - \left[A^* \frac{\partial d_A}{\partial x} + B^* \frac{\partial d_B}{\partial x} \right] = 0 \end{aligned} \quad (53)$$

The equation (53) is written as follows.

$$\begin{aligned} \frac{\partial \pi}{\partial x} = & -p \frac{\partial d_z}{\partial x} - \left[(p_A + d_A) \frac{\partial A^*}{\partial Q_z} \cdot \frac{\partial Q_z}{\partial x} + (p_B + d_B) \frac{\partial B^*}{\partial Q_z} \cdot \frac{\partial Q_z}{\partial x} \right] \\ & - \left[A^* \frac{\partial d_A}{\partial x} + B^* \frac{\partial d_B}{\partial x} \right] = 0 \end{aligned} \quad (54)$$

In a similar manner, $\frac{d\pi}{dy}$ is given as

$$\begin{aligned} \frac{\partial \pi}{\partial y} = & -p \frac{\partial d_z}{\partial y} - \left[(p_A + d_A) \frac{\partial A^*}{\partial Q_z} \cdot \frac{\partial Q_z}{\partial y} + (p_B + d_B) \frac{\partial B^*}{\partial Q_z} \cdot \frac{\partial Q_z}{\partial y} \right] \\ & - \left[A^* \frac{\partial d_A}{\partial y} + B^* \frac{\partial d_B}{\partial x} \right] = 0 \end{aligned} \quad (55)$$

Moreover, $\partial \pi / \partial p$ is shown as

$$\begin{aligned} \frac{\partial \pi}{\partial p} = & K - 2p - d_z + \left[\frac{\alpha}{\beta} (p_A + d_A)^\alpha (p_B + d_B)^\beta \right. \\ & \left. + \frac{\beta}{\alpha} (p_A + d_A)^\alpha \cdot (p_B + d_B)^\beta \right] = 0 \end{aligned} \quad (56)$$

d_z , d_A , and d_B in each equation is shown, respectively, as a function of x and y as in the following.

$$d_z = \sqrt{x^2 + y^2} \quad (57)$$

$$d_A = \sqrt{(x - C)^2 + y^2} \quad (58)$$

$$d_B = \sqrt{(x - W)^2 + (y - V)^2} \quad (59)$$

Now, the simultaneous equations of (54), (55) and (56) are solved by numerical analysis.⁹⁾ Suppose that $\alpha = \beta = 0.5$, $p_A = p_B = 9$, $K = 250$, $C = 9$, $W = 5$, $V = 9$, and solve for optimal price p^* and optimal location (X^*, Y^*) . The solutions are: $p^* = 136.76$, $X^* = 5.18$, and $Y^* = 2.15$. Therefore, $A^* = 117.12$, $B^* = 98.91$, $Q_z = 107.63$. Of course, if values of α , β , p_A and p_B vary, optimal location would shift, and there can be a case in which optimal location coincides with material-producing site. Furthermore, if production function changes, market can coincide with optimal location.¹⁰⁾ Hence, it follows that a profit-maximizing point is not a priori limited to a particular point.

IV Summary and Conclusion

Conclusions set forth in our analysis above can be summarized as follows. In a linear market, a firm's pricing behavior is different according to whether market length is given or is limited. If the former case is presupposed, a firm's optimal price, aggregate sales, revenue, cost and profit would vary according to a firm's location shift. Manners of change of these variables are not uniform. Particularly, aggregate sales and profit can be both of a concave and of a convex function. A factor which would influence a type of this function is, a production function remaining constant, level of material price for market length. For example, when two material prices are assumed to be equal, and these prices go below a given level, a firm's profit is described as a convex curve. Profit is maximized at end points of a linear space of a material-producing site. Namely, a firm's location will coincide with a material-producing site. On the contrary, when p_A and p_B get higher, a firm will locate itself at a center of a linear space.

In analysis of location in a plane space, numerical analysis has derived a firm's optimal location and optimal price. A firm's location depends upon various variables, and is not limited to a material-producing site, a simple weight of location geometry, or a market place. Here, we did not analyze how aggregate sales, revenue, cost and profit would change as a firm shifts its location. A critical analysis of this problem will remain a topic of our future research.

Notes and References

- 1) Ishikawa, T. "Yusohi Saishoka to Keiei Ritchi" (transport-cost minimization and Firm's Location), *Kokusai Kenkyu*, No. 4, 1987, pp.63-76.

- 2) Ishikawa, T. and Toda, M "Production Cost Minimization and Location of Firm", *Kokusai Kenkyu*, No. 5, 1988, pp.143-63.
- 3) Sakashita, N. "Production Function, Demand Function and Location Theory of the Firm", *Papers of Regional Association*, XX, 1968, pp. 109-122.
- 4) Ibid. 2), pp.144-152.
- 5) Take a note that a minimum point V appears within the scope of market 1. These maximum and minimum points can be obtained by equating the equation (44) to zero and solving it for x .
- 6) For example, even if a material price remains constant, a decrease of freight rate causes materials prices p_A and p_B to rise relative to transportation cost for distances 1 and $(1 - x)$. Based on this fact, a firm can maximize its profit by fixing its location at a center of a linear space. On the contrary, if a freight rate goes up sufficiently, a firm's location is going to move to end points of a linear space.
- 7) Here, we will derive only an optimal location point. It will not be considered how revenue, cost and profit would vary according to a firm's shift of location.
- 8) As for a transformation from the equation (53) to the equation (54), see the following article. Shieh, Y.N., "The Moses-Predohl pull and neoclassical location theory," *Regional Science and Urban Economics*, 13, 1983, pp.517-524, Silberberg, *The Structure of Economics*, New York, 1978, Chap.6.
- 9) See *ibid.* 1), pp.67-68 for a method of numerical analysis.
- 10) For example, when γ of production function shown in the equation (16) increases, there will arise a possibility that market area becomes a firm's optimal location. See Kahalili, A., Mathur, V.K. and Bodenhorn, D., "Location and the Theory of Production: A Generalization," *Journal of Economic Theory*, 9, 1974, pp.467-75.