

The Number of 2-matchings and Distinct 4-cycles in a Hyperoctahedral Graph

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Abstract

We study the number of 2-matchings and distinct 4-cycles in a hyperoctahedral graph through coefficients of the characteristic polynomial. Also, we consider the chromatic number and the edge chromatic number of a hyperoctahedral graph.

[Keywords] hyperoctahedral graph, 2-matching, 4-cycle, chromatic number

A complete graph K_n is the graph with n vertices in which each distinct pair are adjacent. A graph H_s obtained by removing s disjoint edges from K_{2s} . A graph H_s is sometimes known as a hyperoctahedral graph of s dimensions. This graph is a complete multipartite graph $K_{2,2,2,\dots,2}$. Let $A(H_s)$ be the adjacent matrix of H_s .

Also, let

$$f(\lambda, H_s) = |\lambda E - A(H_s)| = \lambda^{2s} + C_1 \lambda^{2s-1} + C_2 \lambda^{2s-2} + C_3 \lambda^{2s-3} + C_4 \lambda^{2s-4} + \dots$$

be the characteristic polynomial of H_s , where E is an identity matrix. It is well known that $C_1=0$, $-C_2$ equals the size of H_s and $-C_3$ equals twice number of triangles (see, for example [1, N. Biggs, p. 8 1993]).

We recall that the following definition.

Definition 1 (for example [2, Y. Jin and M. Kanemitsu 2007]). Let G be a graph and let M be a subset of the edges set of G . Then M is called a 2-matching if M consists two element e, f and e, f is a property that e and f are independent, that is, e and f are not incident each other.

Let n_M, n_C be the number of 2-matchings, distinct 4-cycles, respectively (for example [2, Y. Jin and M. Kanemitsu 2007] or [4, M. Kanemitsu to appear]).

Lemma 1 ([1, N. Biggs, p. 46 1993]). *If an undirected graph G has no loops, then $C_4 = n_M - 2n_C$.*

Lemma 2 ([2, Y. Jin and M. Kanemitsu, 2007]). *Let G be a simple graph having the order n and let $(d_1, d_2, d_3, \dots, d_n)$ be the degree sequence of G . Then the number n_M of 2-matchings of G equals to the following*

$$\frac{1}{8} \left(\sum_{i=1}^n d_i \right)^2 - \frac{1}{2} \sum_{i=1}^n d_i^2 + \frac{1}{4} \sum_{i=1}^n d_i$$

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where d_i is a degree of a vertex v_i ($i=1, 2, \dots, n$) of G .

Example 1. The hyperoctahedral $H_2=K_{2,2}$ of 2 dimensions has two 2-matchings and only one distinct 4-cycles. From the adjacent matrix $A(H_2)$, we obtain that

$$f(\lambda, H_2)=\lambda^4-4\lambda^2=\lambda^2(\lambda+2)(\lambda-2).$$

$$C_1=0, -C_2=4, n_M=2, n_C=1 \text{ and } C_4=n_M-2n_C=0.$$

The degree sequence of the graph H_2 is the sequence $(2, 2, 2, 2)$. By the above formula, we also obtain that

$$n_M=\frac{1}{8}(2+2+2+2)^2-\frac{1}{2}(2^2+2^2+2^2+2^2)+\frac{1}{4}(2+2+2+2)=2.$$

Therefore $0=C_4=2-2n_C$. Hence $n_C=1$.

The (vertex) chromatic number of a graph G is denoted by $\chi(G)$ (see, for example [3, KJ 2002]). Also, the edges chromatic number of a graph G is denoted by $\chi'(G)$. Then $\chi(H_2)=2$ and $\chi'(H_2)=2$. This graph H_2 has no triangle.

Example 2. The hyperoctahedral $H_3=K_{2,2,2}$ of 3 dimensions has six vertices and twelve edges (that is, the order of H_3 is 6 and the size of H_3 is 12). In this graph, there exists 8 triangles. The degree sequence of the hyperoctahedral of 3 dimensions is the sequence $(4, 4, 4, 4, 4, 4)$. From the adjacent matrix $A(H_3)$, we obtain that

$$f(\lambda, H_3)=\lambda^6-12\lambda^4+16\lambda^3=\lambda^3(\lambda-4)(\lambda+2)^2.$$

We have $n_M=30$, $n_C=15$, $\chi(H_3)=3$ =the number of index 2 in $H_3=K_{2,2,2}$ and $\chi'(H_3)=4=\deg(a)$ for each vertex a .

Example 3. The hyperoctahedral $H_4=K_{2,2,2,2}$ of 4 dimensions has 8 vertices and 24 edges (that is, the order of H_4 is 8 and the size of H_4 is 24). In this graph, there exists 32 triangles. The degree sequence of the hyperoctahedral of 4 dimensions is the sequence $(6, 6, 6, 6, 6, 6, 6, 6)$. From the adjacent matrix $A(H_4)$, we obtain that

$$f(\lambda, H_4)=\lambda^8-24\lambda^6+64\lambda^5-48\lambda^4=\lambda^4(\lambda-6)(\lambda+2)^3.$$

We have $n_M=156$, $n_C=102$, $\chi(H_4)=4$ =the number of index 2 in $H_4=K_{2,2,2,2}$ and $\chi'(H_4)=6=\deg(a)$ for each vertex a .

Example 4. The hyperoctahedral $H_5=K_{2,2,2,2,2}$ of 5 dimensions has 10 vertices and 40 edges (that is, the order of H_5 is 10 and the size of H_5 is 40). In this graph, there exists 80 triangles. The degree sequence of the hyperoctahedral of 5 dimensions is the sequence $(8, 8, 8, 8, 8, 8, 8, 8, 8, 8)$. From the adjacent matrix $A(H_5)$, we obtain that

$$f(\lambda, H_5)=\lambda^{10}-40\lambda^8+160\lambda^7-240\lambda^6+128\lambda^5=\lambda^5(\lambda-8)(\lambda+2)^4.$$

We have $n_M=500$, $n_C=370$, $\chi(H_5)=5$ =the number of index 2 in $H_5=K_{2,2,2,2,2}$ and $\chi'(H_5)=8=\deg(a)$ for each vertex a .

In general, we have the following results.

Theorem 3. Let H_s be a hyperoctahedral graph with s dimensions. Then the following statements hold.

- (1) [1, N. Biggs 1993] *The characteristic polynomial $f(\lambda, H_s)$ of a hyperoctahedral graph H_s with s dimensions is the following form.*

$$f(\lambda, H_s) = \lambda^s(\lambda - 2(s-1))(\lambda + 2)^{s-1} = \lambda^{2s} - 2s(s-1)\lambda^{2s-2} - \frac{8}{3}s(s-1)(s-2)\lambda^{2s-3} \\ - 2s(s-1)(s-2)(s-3)\lambda^{2s-4} - \dots + \{2^{r_{s-1}}C_r - 2^r(s-1)_{s-1}C_{r-1}\}\lambda^{2s-r} + \dots$$

- (2) *The order of a hyperoctahedral H_s with s dimensions is $2s$ and the size of this graph is $2s(s-1)$.*

- (3) *The number of the triangles in H_s is $\frac{4}{3}s(s-1)(s-2)$.*

- (4) *The number n_M of 2-matchings in H_s is the following*

$$n_M = s(s-1)(2s^2 - 6s + 5)$$

- (5) *The number n_C of distinct 4-cycles is the following.*

$$n_C = \frac{1}{2}s(s-1)(4s^2 - 16s + 17).$$

- (6) *Let $\chi(H_s)$ be the chromatic number of H_s and let $\chi'(H_s)$ be the edge chromatic number of H_s . Then $\chi(H_s) = s$ and $\chi'(H_s) = \text{degree}(v)$ for each vertex v of H_s .*

[Proof]

- (1) The adjacent matrix $A(H_s)$ of a hyperoctahedral of s dimensions is a circulant matrix, that is, we may take the first row of $A(H_s)$ to be $[a_1, a_2, \dots, a_{2s}]$, where each entry is 1, except that $a_1 = a_{s+1} = 0$. In this case, it is well known that the characteristic polynomial $f(\lambda, H_s)$ equals to

$$\lambda^s(\lambda - 2(s-1))(\lambda + 2)^{s-1}.$$

- (2) Since H_s obtained by removing s disjoint edges from the complete graph K_{2s} , we have that the order of H_s equals to the order $2s$ of K_{2s} and the size of H_s equals to a half of the size of K_{2s} . Then the size of K_{2s} equals

$${}_{2s}C_2 - s = \frac{2s(2s-1)}{2!} - s = 2s(s-1).$$

- (3) Since $-C_3 = 2 \times (\text{number of triangles})$, we have that the number of triangles equals to $\frac{4}{3}s(s-1)(s-2)$ by (1).

- (4) The degree sequence of H_s is $(2(s-1), 2(s-1), 2(s-1), \dots, 2(s-1))$. By Lemma 2, we have that

$$n_M = \frac{1}{8} \times \{2s(2(s-1))\}^2 - \frac{1}{2} \times 2s\{2(s-1)\} + \frac{1}{4} \times 2s(2(s-1)) = s(s-1)(2s^2 - 6s + 5).$$

- (5) By (1), we have that $C_4 = 2s(s-1)(s-2)(s-3)$. By Lemma 1, $C_4 = n_M - 2n_C$. Using (4), we have that

$$n_C = \frac{1}{2}(n_M - C_4) = \frac{1}{2}(s(s-1)(2s^2 - 6s + 5) - 2s(s-1)(s-2)(s-3)) = s(s-1)(2s^2 - 6s + 5).$$

- (6) By the complete multipartite graph, $\chi(H_s) = s$, $\chi'(H_s) = 2s - 2 = 2(s-1) = \text{degree}(v)$ for each vertex v of H_s .

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