

A stronger version of stationarity preserved under $<\kappa$ -strategically closed forcing

Sakaé FUCHINO*

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We introduce a normal filter $\mathcal{F}_\kappa^{SIA}\lambda$ over $\mathcal{P}_\kappa\lambda$ for an inaccessible cardinal κ and $\lambda \geq \kappa$, and show that $\mathcal{F}_\kappa^{SIA}\lambda$ -stationarity of a subset of $\{x \in \mathcal{P}_\kappa\lambda : x \cap \kappa \text{ is a singular ordinal}\}$ is preserved in generic extensions by $<\kappa$ -strategically closed partial ordering (Theorem 1). In particular, the stationarity of such a set is preserved in these generic extensions. We also give a proof of a theorem by T. Usuba which shows that $\mathcal{F}_\kappa^{SIA}\lambda$ -stationarity in Theorem 1 cannot be replaced by stationarity (Theorem 3). This note complements [2] where Theorem 1 and Theorem 3 are cited without proof.

Keywords : stationary sets, internally approachable models, club shooting, $\mathcal{P}_\kappa\lambda$

1 Introduction

The main theorem of this note is the following:

Theorem 1 *Suppose that κ is an inaccessible cardinal and λ is a cardinal with $\lambda \geq \kappa$. If*

$$(1.1) \quad S \subseteq \{x \in \mathcal{P}_\kappa\lambda : x \cap \kappa \text{ is a singular ordinal}\}$$

and S is $\mathcal{F}_\kappa^{SIA}\lambda$ -stationary, then, for any $<\kappa$ -strategically closed p.o.-set $\mathbb{P}$, we have:

$$\Vdash_{\mathbb{P}} "S \text{ is } \mathcal{F}_\kappa^{SIA}\lambda\text{-stationary}."$$

In particular, S remains stationary in any generic extension by such $\mathbb{P}$.

Here, $\mathcal{F}_\kappa^{SIA}\lambda$ is a normal filter defined in the next section and $<\kappa$ -strategically closedness of a p.o.-set is the weakening of $<\kappa$ -closedness defined below.

In Theorem 4 of [2], a variant of Theorem 1 above is proved in which $\mathcal{F}_\kappa^{SIA}\lambda$ -stationarity is slightly weakened to be what is called there $\mathcal{F}_\kappa^{IA}\lambda$ -stationarity, κ may be just regular and S may be any $\mathcal{F}_\kappa^{IA}\lambda$ -stationary subset of $\mathcal{P}_\kappa\lambda$ but only preservation under generic extension by κ -strategically closed p.o.-set is

concluded. The proof of Theorem 1 is a slight modification of the proof of this theorem. Here, a p.o.-set $\mathbb{P}$ is said to be κ -strategically closed if the player Nonempty has a winning strategy in the following infinitary game $\mathcal{G}_\kappa^{\text{II}}(\mathbb{P})$ (see e.g. [3], [12]):

$\mathcal{G}_\kappa^{\text{II}}(\mathbb{P})$ is a game over $\mathbb{P}$ between two players Empty and Nonempty. The player Empty begins the game with his move $p_0 \in \mathbb{P}$. At $\alpha = 1 + \xi + 2n + 1$ st move for a limit ordinal $\xi < \kappa$ (or $\xi = 0$) and $n \in \omega$, the player Empty plays $p_\alpha \in \mathbb{P}$; Nonempty plays $p_\alpha \in \mathbb{P}$ at $\alpha = 1 + \xi + 2n$ th move; each p_α must be below the previous moves p_ξ , $\xi < \alpha$ (with respect to $\leq_{\mathbb{P}}$) so that p_ξ , $\xi < \alpha$ form a decreasing sequence in $\mathbb{P}$ for all $\alpha < \kappa$. Nonempty wins if the game can be played through all the moves p_ξ , $\xi < \kappa$. $\mathbb{P}$ is said to be strongly κ -strategically closed if Nonempty has a winning strategy in a game $\mathcal{G}_\kappa^{\text{I}}(\mathbb{P})$ which is similar to $\mathcal{G}_\kappa^{\text{II}}(\mathbb{P})$ but Empty plays at $\xi + 2n$ th move and Nonempty at $\xi + 2n + 1$ st for any limit ordinal ξ (or $\xi = 0$) and $n \in \omega$.

A p.o.-set $\mathbb{P}$ is said to be (strongly) $<\kappa$ -strategically closed if $\mathbb{P}$ is (strongly) λ -strategically closed for all $\lambda < \kappa$.

*Department of Natural Science and Mathematics:

A game in $\mathcal{G}_\kappa^{\text{II}}(\mathbb{P})$ (or in $\mathcal{G}_\kappa^{\text{I}}(\mathbb{P})$) is also represented as

$$p_0 \geq_{\mathbb{P}} q_0 \geq_{\mathbb{P}} p_1 \geq_{\mathbb{P}} q_1 \cdots \geq_{\mathbb{P}} q_\alpha \geq_{\mathbb{P}} p_\alpha \cdots$$

(or as $p_0 \geq_{\mathbb{P}} q_0 \geq_{\mathbb{P}} p_1 \geq_{\mathbb{P}} q_1 \cdots \geq_{\mathbb{P}} p_\alpha \geq_{\mathbb{P}} q_\alpha \cdots$ respectively). where the p_α 's and q_α 's are moves taken by Empty and Nonempty respectively.

The following Theorem 2 shows that the condition (1.1) cannot be removed from Theorem 1 while Theorem 3 shows that $\mathcal{F}_\kappa^{\text{SIA}}\lambda$ -stationarity in Theorem 1 cannot be replaced by stationarity.

Theorem 2 Suppose that κ is Mahlo and $\lambda > \kappa$ is regular. Then

$$S = \{x \in \mathcal{P}_\kappa \lambda : x \cap \kappa \text{ is a regular cardinal}\}$$

is $\mathcal{F}_\kappa^{\text{SIA}}\lambda$ -stationary but there is a $<\kappa$ -strategically closed κ^+ -c.c. p.o.-set $\mathbb{P}$ such that

$$\Vdash_{\mathbb{P}} \text{"} S \text{ is non-stationary"}$$

Theorem 3 (T. Usuba [10, 11]) Suppose that κ is weakly inaccessible with $\kappa^{<\kappa} = \kappa$ and $\lambda > \kappa$ is regular. Let

$$S_0 = \{x \in \mathcal{P}_\kappa \lambda : \text{cf}(\sup(x)) > x \cap \kappa, \\ x \text{ is stationary in } \sup x\}.$$

Then there is a κ -strategically closed κ^+ -c.c. p.o.-set $\mathbb{P}$ such that $\Vdash_{\mathbb{P}} \text{"} S_0 \text{ is non-stationary subset of } \mathcal{P}_\kappa \lambda \text{"}$.

Note that S in Theorem 3 is stationary if κ is $\lambda^{<\kappa}$ -supercompact (see Lemma 7). It is easily seen that this S can never be $\mathcal{F}_\kappa^{\text{SIA}}\lambda$ -stationary.

The set-theoretic notation and terminology in this note are quite standard. The reader should consult [7] or [4] for undefined notions. A partial ordering used to build generic extension is called here a p.o.-set. The forcing is denoted in reverse Jerusalem notation. That is, smaller conditions are stronger. On the other hand, in place of the dotted symbols $\dot{a}$, $\dot{b}$, etc. used in [4] and [7], we make use of symbols with a tilde below them, like $\underset{\sim}{a}$, $\underset{\sim}{b}$, etc. to denote $\mathbb{P}$ -names for a p.o.-set $\mathbb{P}$.

2 Preservation of $\mathcal{F}_\kappa^{\text{SIA}}\lambda$ -stationarity

Let κ be an inaccessible cardinal and $\kappa < \lambda$. In this section we show that $\mathcal{F}_\kappa^{\text{SIA}}\lambda$ -stationarity defined below is indestructible in generic extensions by $<\kappa$ -strategically $<\kappa$ -closed p.o.-sets.

Let μ be sufficiently large (regular) cardinal. For $a \in \mathcal{H}(\mu)$, let

$$S_\kappa^{\text{SIA } \mu, a} \lambda = \{\lambda \cap M : M \prec \mathcal{H}(\mu), \kappa, \lambda, a \in M, M \text{ is} \\ \text{strongly internally approachable,} \\ |M| < \kappa, \kappa \cap M \in \kappa\}$$

Here, an elementary submodel M of $\mathcal{H}(\mu)$ is said to be *strongly internally approachable* if there is an increasing sequence $\langle M_\alpha : \alpha < \delta \rangle$ of elementary submodels of M such that

$$(2.1) \quad \bigcup_{\alpha < \delta} M_\alpha = M,$$

$$(2.2) \quad \langle M_\alpha : \alpha < \gamma \rangle \in M_\gamma \text{ for all } \gamma < \delta \text{ and}$$

$$(2.3) \quad \mathcal{P}(\gamma) \in M_\gamma \text{ for all limit } \gamma < \delta.$$

$\langle M_\alpha : \alpha < \delta \rangle$ as above is said to be a *witness* of the strong internal approachability of M . Let

$$\mathcal{F} = \{S_\kappa^{\text{SIA } \mu, a} \lambda : a \in \mathcal{H}(\mu), \mu \text{ is sufficiently large}\}$$

Lemma 4 (a) $\mathcal{F}$ has the finite intersection property.

(b) If $a \in \mathcal{H}(\mu)$ and $\mu < \mu'$ then

$$S_\kappa^{\text{SIA } \mu', \langle a, \mu \rangle} \lambda \subseteq S_\kappa^{\text{SIA } \mu, a} \lambda.$$

Proof. (a): Suppose that $\mu_0, \dots, \mu_{n-1}$ are all sufficiently large and $a_0 \in \mathcal{H}(\mu_0), \dots, a_{n-1} \in \mathcal{H}(\mu_{n-1})$. Let μ be sufficiently large compared to $\mu_0, \dots, \mu_{n-1}$. Let $a = \langle \langle a_i : i < n \rangle, \langle \mu_i : i < n \rangle \rangle$. Then $S_\kappa^{\text{SIA } \mu, a} \lambda \subseteq S_\kappa^{\text{SIA } \mu_i, a_i} \lambda$ for all $i < n$ (for details see the proof of Claim 5.1). Hence

$$\bigcap_{i < n} S_\kappa^{\text{SIA } \mu_i, a_i} \lambda \neq \emptyset.$$

(b): Suppose that $x \in S_\kappa^{\text{SIA } \mu', \langle a, \mu \rangle} \lambda$.

Then there is a strongly internally approachable $M \prec \mathcal{H}(\mu')$ such that $\kappa, \lambda, \langle a, \mu \rangle \in M, |M| < \kappa, \kappa \cap M \in \kappa$ and $x = \lambda \cap M$. Let $\langle M_\alpha : \alpha < \delta \rangle$ be a witness of strong internal approachability of M . Without loss of generality we may assume that $\mu \in M_0$. $M' \prec \mathcal{H}(\mu_\alpha)$ by $\mu_\alpha \in M$ and $\langle M_\alpha \cap \mathcal{H}(\mu) : \alpha < \delta \rangle$ is a witness of strong internal approachability of $M^* = M \cap \mathcal{H}(\mu) \prec \mathcal{H}(\mu)$. Since $x = \lambda \cap M = \lambda \cap M^*$ and $a \in M^*$, it follows that $x \in S_\kappa^{\text{SIA } \mu, a} \lambda$. $\square$ (Lemma 4)

Let

$$\mathcal{F}_\kappa^{\text{SIA}} \lambda = \text{the filter on } \mathcal{P}_\kappa \lambda \text{ generated from } \mathcal{F}.$$

Lemma 5 (a) $S_\kappa^{SIA \mu, a} \lambda$ is a stationary subset of $\mathcal{P}_\kappa \lambda$ for any $a \in \mathcal{H}(\mu)$.

(b) $\mathcal{F}_\kappa^{SIA} \lambda$ is a normal filter on $\mathcal{P}_\kappa \lambda$ (in the sense of [6]). In particular, $\mathcal{F}_\kappa^{SIA} \lambda$ includes the club filter on $\mathcal{P}_\kappa \lambda$.

Proof. (a) follows from (b).

For (b), we have to show that

$$(2.4) \quad \mathcal{F}_\kappa^{SIA} \lambda \text{ is a fine filter;}$$

$$(2.5) \quad \mathcal{F}_\kappa^{SIA} \lambda \text{ is } < \kappa\text{-complete;}$$

$$(2.6) \quad \mathcal{F}_\kappa^{SIA} \lambda \text{ is closed under diagonal intersection.}$$

(2.4) is clear from the definition of $\mathcal{F}_\kappa^{SIA} \lambda$. We prove here (2.6). (2.5) can be shown similarly.

Suppose that $X_\alpha \in \mathcal{F}_\kappa^{SIA} \lambda$ for $\alpha \in \lambda$. We may assume that $X_\alpha = S_\kappa^{SIA \mu_\alpha, a_\alpha} \lambda$ for some sufficiently large μ_α and $a_\alpha \in \mathcal{H}(\mu_\alpha)$. Let μ be sufficiently large compared to $\sup\{\mu_\alpha : \alpha \in \lambda\}$ and

$$a = \{\langle \mu_\alpha : \alpha < \lambda \rangle, \langle a_\alpha : \alpha < \lambda \rangle\}.$$

Then we have

Claim 5.1

$$\begin{aligned} \Delta_{\alpha < \lambda} X_\alpha &= \{x \in \mathcal{P}_\kappa \lambda : (\forall \alpha \in x)(x \in X_\alpha)\} \\ &\supseteq S_\kappa^{SIA \mu, a} \lambda. \end{aligned}$$

⊢ Suppose that $x \in S_\kappa^{SIA \mu, a} \lambda$. We show that $x \in \Delta_{\alpha < \lambda} X_\alpha$. Let $M \prec \mathcal{H}(\mu)$ be such that $x = \lambda \cap M$, $\kappa, \lambda, a \in M$, M is internally approachable, $|M| < \kappa$ and $\kappa \cap M \in \kappa$. Let $\langle M_\xi : \xi < \delta \rangle$ be a witness of strong internal approachability of M .

If $\alpha \in x = \lambda \cap M$, then $\mu_\alpha, a_\alpha \in M$. Let $M' = M \cap \mathcal{H}(\mu_\alpha)$. Then $\langle M_\xi \cap \mathcal{H}(\mu_\alpha) : \xi < \delta \rangle$ witnesses that M' is internally approachable. Clearly we have $x = \lambda \cap M'$, $\kappa, \lambda, a \in M'$ and $|M'| < \kappa$. Hence $x \in S_\kappa^{SIA \mu_\alpha, a_\alpha} \lambda = X_\alpha$. ⊣ (Claim 5.1)

It follows that $\Delta_{\alpha < \lambda} X_\alpha \in \mathcal{F}_\kappa^{SIA} \lambda$. □ (Lemma 5)

Let us call $S \subseteq \mathcal{P}_\kappa \lambda$ an $\mathcal{F}_\kappa^{SIA} \lambda$ -stationary subset of $\mathcal{P}_\kappa \lambda$ if $S \cap F \neq \emptyset$ for all $F \in \mathcal{F}_\kappa^{SIA} \lambda$. By Lemma 5, (b) $\mathcal{F}_\kappa^{SIA} \lambda$ -stationary subset of $\mathcal{P}_\kappa \lambda$ is stationary.

Proof of Theorem 1: By (1.1), the mapping $S \ni x \mapsto \text{cf}(x \cap \kappa)$ is a well defined regressive function. Thus, by Lemma 5, (b), we may assume that there is a $\nu < \kappa$ such that $\text{cf}(x \cap \kappa) = \nu$ for all $x \in S$.

Let σ_ξ , $\xi < \kappa$ be a sequence of strategies of the player Nonempty in the games $\mathcal{G}_\xi^{\text{II}}(\mathbb{P})$, $\xi < \kappa$.

Let μ be sufficiently large. Suppose that $p \in \mathbb{P}$ and $\tilde{a} \in \mathcal{H}(\mu)$ is a $\mathbb{P}$ -name of an element of $\mathcal{H}(\mu)^{V[G]}$ for some sufficiently large μ . We show that there is a $q \leq_{\mathbb{P}} p$ such that

$$q \Vdash_{\mathbb{P}} "S \cap S_\kappa^{SIA \mu, \tilde{a}} \lambda \neq \emptyset".$$

Let μ' be sufficiently large compared to μ and let $\leq_{\mathcal{H}(\mu)} \in \mathcal{H}(\mu')$ be a well-ordering of $\mathcal{H}(\mu)$. Let $M \prec \mathcal{H}(\mu')$ be strongly internally approachable such that

$$(2.7) \quad \kappa, \lambda, \mu, \mathbb{P}, \leq_{\mathcal{H}(\mu)}, p, \tilde{a}, \langle \sigma_\xi : \xi < \kappa \rangle, \sigma_\nu \in M,$$

$$(2.8) \quad \kappa \cap M \in \kappa, |M| < \kappa \text{ and } \lambda \cap M \in S.$$

This is possible by $\mathcal{F}_\kappa^{SIA} \lambda$ -stationarity of S .

Let $\langle M_\alpha : \alpha < \delta \rangle$ be a witness of strong internal approachability of M . Without loss of generality we may assume that $\kappa, \lambda, \mu, \mathbb{P}, \leq_{\mathcal{H}(\mu)}, p, \tilde{a}, \sigma \in M_0$.

Let $M^* = M \cap \mathcal{H}(\mu)$ and $M_\alpha^* = M_\alpha \cap \mathcal{H}(\mu)$ for $\alpha < \delta$. Note that

$$(2.9) \quad M^* \prec \mathcal{H}(\mu) \text{ and the sequence } \langle M_\alpha^* : \alpha < \delta \rangle \text{ witnesses that } M^* \text{ is strongly internally approachable.}$$

We also have $\lambda \cap M^* = \lambda \cap M \in S$, $\lambda \cap M_\alpha^* = \lambda \cap M_\alpha$ for all $\alpha < \delta$ and

$$(2.10) \quad \kappa, \lambda, \mu, \mathbb{P}, \leq_{\mathcal{H}(\mu)}, p, \tilde{a}, \langle \sigma_\xi : \xi < \kappa \rangle, \sigma_\delta \in M^*,$$

$$(2.11) \quad \kappa \cap M^* \in \kappa, |M^*| < \kappa \text{ and } \lambda \cap M^* \in S.$$

Note that

$$(2.12) \quad \text{cf}(\delta) = \text{cf}(\kappa \cap M) = \text{cf}(\lambda \cap M) = \nu.$$

Hence there is a continuously increasing sequence δ_ξ , $\xi < \nu$ of ordinals cofinal in δ . Let

$$p_0 \geq_{\mathbb{P}} q_0 \geq_{\mathbb{P}} p_1 \geq_{\mathbb{P}} q_1 \geq_{\mathbb{P}} \cdots \geq_{\mathbb{P}} q_\alpha \geq_{\mathbb{P}} p_\alpha \geq_{\mathbb{P}} \cdots$$

for $\alpha < \nu$ be a play in the game $\mathcal{G}_\nu^{\text{II}}(\mathbb{P})$ such that

$$(2.13) \quad p_0 \leq_{\mathbb{P}} p;$$

$$(2.14) \quad \text{each } q_\alpha \text{ is chosen according to } \sigma_\nu;$$

$$(2.15) \quad \langle p_\beta, q_\beta : \beta < \alpha \rangle \in M_{\delta_{\alpha+1}}^*;$$

$$(2.16) \quad \text{For every } \mathbb{P}\text{-name } \beta \in M_{\delta_\alpha}^* \text{ of element of } \lambda, \text{ there is } \tilde{\beta} \in \lambda \cap M_{\delta_{\alpha+1}}^* \text{ such that } p_\alpha \Vdash_{\mathbb{P}} " \tilde{\beta} = \beta "; p_\alpha \text{ is minimal with respect to } \leq_{\mathcal{H}(\mu)} \text{ with this property.}$$

We can find such p_α as in (2.16) (possibly without minimality) by playing a $\mathcal{G}_\tau^{\text{II}}(\mathbb{P})$ -game in $M_{\delta_{\alpha+1}}^*$ for $\tau = |M_\alpha^*|$ so that Empty's moves fix $\mathbb{P}$ -names $\in M_{\delta_\alpha}$ of ordinals in λ one by one while the player Nonempty plays according to $\sigma_\tau \in M_{\delta_{\alpha+1}}^*$. (A candidate for) p_α can be then taken as an element in $\mathbb{P}$ below this play which exists because of σ_τ .

The whole construction is now possible, since by the clause on minimality in (2.16), the sequence $\langle p_\beta, q_\beta : \beta < \alpha \rangle$ is definable in $M_{\delta_{\alpha+1}}^*$. Note that

$$\langle \delta_\beta : \beta \leq \alpha \rangle, \leq_{\mathcal{H}(\mu)} \cap (M_{\delta_\alpha}^* \times M_{\delta_\alpha}^*) \in M_{\delta_{\alpha+1}}^*.$$

Thus, these sets are available as parameters in the definition of $\langle p_\beta, q_\beta : \beta < \alpha \rangle$ in $M_{\delta_{\alpha+1}}^*$. Hence (2.15) holds for all $\alpha < \nu$.

By (2.14), there is $q \leq_{\mathbb{P}} q_\alpha$ for all $\alpha < \nu$. By (2.16), we have

$$q \Vdash_{\mathbb{P}} \text{"}\lambda \cap M^* = \lambda \cap M^*[\tilde{G}]\text{"}.$$

On the other hand, $\Vdash_{\mathbb{P}} \text{"}\lambda \cap M^*[\tilde{G}] \in S_\kappa^{\text{SIA } \mu, a} \lambda \text{"}$ by (2.9) and (2.10). It follows that

$$q \Vdash_{\mathbb{P}} \text{"}S \cap S_\kappa^{\text{SIA } \mu, a} \lambda \neq \emptyset\text{"}$$

as desired. $\square$ (Theorem 1)

3 Shooting clubs through $\mathcal{P}_\kappa \lambda$

In this section we give proofs of Theorem 2 and Theorem 3. Let us begin with Theorem 2.

Suppose that κ is inaccessible and $S^* = \{\alpha \in \kappa : \alpha \text{ is singular}\}$. Let $\mathbb{P}_{S^*}$ be the standard p.o.-set for shooting a club through S^* . That is,

$$\mathbb{P}_{S^*} = \{p \subseteq \kappa : p \text{ is a bounded closed subset of } S^*\}$$

with end-extension as the ordering on $\mathbb{P}_{S^*}$. The following is well-known (see, for example Exercise 23.8 in [4]):

Lemma 6 $\mathbb{P}_{S^*}$ has a $<\mu$ -closed dense subset for all $\delta < \kappa$. In particular, $\mathbb{P}_{S^*}$ is $<\mu$ -strategically closed.

Proof. For a cardinal $\mu < \kappa$, let $\mathbb{P}_{S^*}^\mu = \{p \in \mathbb{P}_{S^*} : \max(p) > \mu\}$.

Clearly $\mathbb{P}_{S^*}^\mu$ is dense in $\mathbb{P}_{S^*}$. $\mathbb{P}_{S^*}^\mu$ is $<\mu$ -closed: Suppose that $\nu < \mu$ and $\langle p_\alpha : \alpha < \mu \rangle$ is a descending sequence in $\mathbb{P}_{S^*}^\mu$. Let $\alpha^* = \sup \bigcup_{\alpha < \nu} p_\alpha$. Then α^* has cofinality $\leq \nu < \mu$ and $\alpha^* \geq \mu$. It follows that

$\alpha^* \in S^*$. Hence $p = \bigcup_{\alpha < \nu} p_\alpha \cup \{\alpha^*\}$ is an element of $\mathbb{P}_{S^*}^\mu$ and $p \leq_{\mathbb{P}_{S^*}^\mu} p_\alpha$ for all $\alpha < \nu$. $\square$ (Lemma 6)

Proof of Theorem 2: Suppose that κ is a Mahlo cardinal and let S be as in the theorem.

Let $\mu > \lambda$ be large enough and $a \in \mathcal{H}(\mu)$. We show that $S \cap S_\kappa^{\text{SIA } \mu, a} \lambda \neq \emptyset$.

Let $\langle M_\alpha : \alpha < \kappa \rangle$ be a continuously increasing sequence of elementary submodels of $\mathcal{H}(\mu)$ such that

$$(3.1) \quad a, S \in M_0;$$

$$(3.2) \quad |M_\alpha| < \kappa \text{ for all } \alpha < \kappa;$$

$$(3.3) \quad M_\alpha \cap \kappa \in \kappa;$$

$$(3.4) \quad \kappa \subseteq \bigcup_{\alpha < \kappa} M_\alpha \text{ and}$$

$$(3.5) \quad \mathcal{P}(M_\alpha) \in M_{\alpha+1}.$$

For $\alpha < \kappa$, let $\beta_\alpha = M_\alpha \cap \kappa$. Then $\langle \beta_\alpha : \alpha < \kappa \rangle$ is a continuously increasing sequence of ordinals cofinal in κ . Since κ is Mahlo, there is an $\alpha^* < \kappa$ such that β_{α^*} is a regular cardinal. Let $x = \lambda \cap M_{\alpha^*}$. Since $x \cap \kappa = M_{\alpha^*} \cap \kappa = \beta_{\alpha^*}$, we have $x \in S$. The sequence $\langle M_{\gamma+1} : \gamma < \alpha^* \rangle$ witnesses the strong internal approachability of M_{α^*} . It follows that $x \in S_\kappa^{\text{SIA } \mu, a} \lambda$.

This shows that S is $\mathcal{F}_\kappa^{\text{SIA } \lambda}$ -stationary.

Let $\mathbb{P} = \mathbb{P}_{S^*}$ as above. Then $\mathbb{P}$ is $<\kappa$ -strategically closed by Lemma 6. $\mathbb{P}$ has the κ^+ -c.c. since $|\mathbb{P}| \leq \kappa$. $\mathbb{P}$ forces $\{x \cap \kappa : x \in S\}$ non-stationary. It follows that $\mathbb{P}$ forces S itself is non-stationary.

$\square$ (Theorem 2)

We now turn to Theorem 3. Let us first show that the set S_0 as in Theorem 3 can be stationary.

Lemma 7 Suppose that $\kappa < \lambda$ and κ is $\lambda^{<\kappa}$ -supercompact. Then

$$S_0 = \{x \in \mathcal{P}_\kappa \lambda : \text{cf}(x) > x \cap \kappa, \\ x \text{ is stationary in } \sup(x)\}$$

is stationary in $\mathcal{P}_\kappa \lambda$.

Proof. Let $\delta = \lambda^{<\kappa}$ and $j : V \rightarrow M$ be an elementary embedding such that

$$(3.6) \quad \text{crit}(j) = \kappa,$$

$$(3.7) \quad j(\kappa) > \delta \text{ and}$$

$$(3.8) \quad {}^\delta M \subseteq M.$$

Suppose that $C \subseteq \mathcal{P}_\kappa \lambda$ is a club set. It is enough to show that $j(S_0) \cap j(C) \neq \emptyset$.

Let $c = \{j(a) : a \in C\}$. Note that $|c| \leq \delta$. Hence, by (3.8), we have $c \in M$. Hence $x = \bigcup c \in M$ as well. Since $M \models c$ is a directed subset of $j(C)$ of cardinality $< j(\kappa)$, we have $M \models x \in j(C)$.

Claim 7.1 $M \models cf(x) = \lambda > \kappa = x \cap j(\kappa)$.

⊢ For $a \in C$, $|a| < \kappa = crit(j)$. Hence $j(a) = \{j(\alpha) : \alpha \in a\}$. Thus $j(a) \cap j(\kappa) = a \cap \kappa$ by (3.6). It follows that $x \cap j(\kappa) = \kappa$. Since $x = \{j(\xi) : \xi < \lambda\}$, we have $M \models cf(x) = \lambda$. $\dashv$ (Claim 7.1)

Claim 7.2 $M \models x$ is stationary in $\sup(x)$.

⊢ Let $\eta = \sup(x)$. Let $D \in M$ be a club subset of η . Then we can find an increasing sequence α_n , $n \in \omega$ of ordinals below η such that

$$(3.9) \quad \alpha_n \in D \text{ if } n \text{ is even};$$

$$(3.10) \quad \alpha_n \in x \text{ if } n \text{ is odd}.$$

For each odd $n \in \omega$, there is an $a_n \in C$ be such that $\alpha_n \in j(a_n)$. Let $\xi_n \in a_n$ be such that $j(\xi_n) = \alpha_n$ and let $\xi = \sup_{n \in \omega} \xi_n$. We have $j(\xi) \in x$ and $j(\xi) = \sup_{n \in \omega} \alpha_n$ by (3.6). Since $\langle \alpha_{2n} : n \in \omega \rangle \in M$ by (3.8) and $\langle \alpha_{2n} : n \in \omega \rangle \subseteq D$, we have $j(\xi) \in D$. $\dashv$ (Claim 7.2)

By Claim 7.1 and Claim 7.2, $M \models x \in j(S_0) \cap j(C)$.

□ (Lemma 7)

Let

$$S_1 = \{x \in \mathcal{P}_\kappa \lambda : |x| > |x \cap \kappa|\}.$$

Lemma 8 $S_0 \subseteq S_1$.

Proof. Suppose that $x \notin S_1$, that is $x \in \mathcal{P}_\kappa \lambda$ and $|x| = |x \cap \kappa|$. If $x \cap \kappa \notin \kappa$ then $x \notin S_0$. If $x \cap \kappa \in \kappa$ then

$$|x| = |x \cap \kappa| \leq x \cap \kappa.$$

It follows $cf(x) \leq |x| \leq x \cap \kappa$. Thus again we have $x \notin S_0$. □ (Lemma 8)

Note that we have $S_\kappa^{SIA} \mu, a \lambda \cap S_1 = \emptyset$ for any sufficiently large μ and $a \in \mathcal{H}(\mu)$. Hence by Lemma 8, S_0 is not $\mathcal{F}_\kappa^{SIA} \lambda$ -stationary.

In the rest, we show that the p.o.-set $\mathbb{P} = \mathcal{Cl}(S_0)$ defined below is as in Theorem 3.

$$(3.11) \quad p \in \mathcal{Cl}(S_0) \Leftrightarrow p : d(p) \times d(p) \rightarrow d(p) \cap \kappa \text{ for some } d(p) \in \mathcal{P}_\kappa \lambda \text{ with } d(p) \cap \kappa \in \kappa \text{ and } cl(p) \cap S_0 = \emptyset$$

where $cl(p) = \{x \in \mathcal{P}(d(p)) : \omega \leq x \cap \kappa < \kappa \text{ and } x \text{ is closed under } p\}$.

For $p, q \in \mathcal{Cl}(S_0)$,

$$(3.12) \quad p \leq_{\mathcal{Cl}(S_0)} q \Leftrightarrow q \subseteq p.$$

Lemma 9 For any $x \in \mathcal{P}_\kappa \lambda$,

$D_x = \{p \in \mathcal{Cl}(S_0) : x \subseteq d(p)\}$ is dense in $\mathcal{Cl}(S_0)$.

Proof. Suppose that $x \in \mathcal{P}_\kappa \lambda$ and $p \in \mathcal{Cl}(S_0)$. Let $y \in \mathcal{P}_\kappa \lambda$ be such that

$$(3.13) \quad y \cap \kappa \in \kappa;$$

$$(3.14) \quad cf(y \cap \kappa) = \omega;$$

$$(3.15) \quad |y| = |y \cap \kappa| \quad (\text{in particular } y \notin S_0 \text{ by Lemma 8});$$

$$(3.16) \quad x \cup d(p) \subseteq y.$$

By (3.14) there is a sequence $\langle \alpha_n : n \in \omega \rangle$ of ordinals cofinal in $y \cap \kappa$.

Let $q : y \times y \rightarrow y \cap \kappa$ be such that

$$(3.17) \quad q \restriction d(p) \times d(p) = p \text{ and}$$

$$(3.18) \quad q(\alpha, n) = \alpha_n \text{ for any } \alpha \in y \setminus d(p) \text{ and } n \in \omega.$$

Claim 9.1 $q \in \mathcal{Cl}(S_0)$.

⊢ We have to show that $cl(q) \cap S_0 = \emptyset$. Suppose that $z \in cl(q)$. Then $z \subseteq d(q) = y$. If $z \subseteq d(p)$ then $z \in cl(p)$ by (3.17). Hence $z \notin S_0$ by the definition (3.11) of $\mathcal{Cl}(S_0)$. Otherwise there is an $\alpha \in z \setminus d(p)$. It follows that $\alpha_n = q(\alpha, n) \in z$ for all $n \in \omega$. Since $z \cap \kappa \in \kappa$ by $z \in cl(p)$ and since α_n , $n \in \omega$ are cofinal in $y \cap \kappa$, we have $z \cap \kappa = y \cap \kappa$. By (3.15), it follows that

$$|z \cap \kappa| \leq |z| \leq |y| \leq |y \cap \kappa| = |z \cap \kappa|.$$

Thus $|z| = |z \cap \kappa|$ and again $z \notin S_0$ by Lemma 8.

$\dashv$ (Claim 9.1)

By the definition (3.12) of $\leq_{\mathcal{Cl}(S_0)}$ and by (3.16), we have $q \leq_{\mathcal{Cl}(S_0)} p$ and $q \in D_x$. □ (Lemma 9)

Lemma 10 $\mathcal{Cl}(S_0)$ has the κ^+ -c.c.

Proof. Suppose that $p_\alpha \in \mathcal{Cl}(S_0)$ for $\alpha \in \kappa^+$. Since κ is inaccessible and $|d(p_\alpha)| < \kappa$ for $\alpha \in \kappa^+$, we may assume that $d(p_\alpha)$, $\alpha \in \kappa^+$ form a Δ -system with root r , $r \supseteq d(p_\alpha) \cap \kappa$ for all $\alpha < \kappa^+$ and $p_\alpha \restriction r \times r$, $\alpha \in \kappa^+$ are all the same.

Fix any $\alpha^* < \beta^* < \kappa^+$. We show that p_{α^*} and p_{β^*} are compatible. Let $y \in \mathcal{P}_\kappa \lambda$ be such that

$$(3.19) \quad y \cap \kappa \in \kappa;$$

$$(3.20) \quad r \cap \kappa \subsetneq y \cap \kappa;$$

$$(3.21) \quad \text{cf}(y \cap \kappa) = \omega;$$

$$(3.22) \quad |y| = |y \cap \kappa| \text{ and}$$

$$(3.23) \quad d(p_{\alpha^*}) \cup d(p_{\beta^*}) \subsetneq y.$$

By (3.20) there is a $\delta^* \in (y \cap \kappa) \setminus r$. Let $\langle \alpha_n : \alpha < \omega \rangle$ be an increasing sequence of ordinals cofinal in $y \cap \kappa$.

Let $q : y \times y \rightarrow y \cap \kappa$ be such that

$$(3.24) \quad q \restriction d(p_{\alpha^*}) \times d(p_{\alpha^*}) = p_{\alpha^*}; \quad q \restriction d(p_{\beta^*}) \times d(p_{\beta^*}) = p_{\beta^*};$$

$$(3.25) \quad q(\xi, \eta) = \delta^* \text{ for all } \langle \xi, \eta \rangle \in (y \setminus d(p_{\alpha^*})) \times (y \setminus d(p_{\beta^*})) \text{ and}$$

$$(3.26) \quad q(\delta^*, n) = \alpha_n \text{ for } n \in \omega.$$

Claim 10.1 $q \in \mathcal{CL}(S_0)$.

— We have to show that $cl(q) \cap S_0 = \emptyset$. Suppose that $z \in cl(q)$. Note that $z \subseteq d(q) = y$. If $z \subseteq d(p_{\alpha^*})$ or $z \subseteq d(p_{\beta^*})$ then $z \in cl(p_{\alpha^*})$ or $z \in cl(p_{\beta^*})$ by (3.24). Hence $z \notin S_0$. Otherwise there are some $\xi \in y \setminus d(p_{\alpha^*})$ and $\eta \in y \setminus d(p_{\beta^*})$. By (3.25), $\delta^* = q(\xi, \eta) \in z$. Hence $\alpha_n \in z$ for all $n \in \omega$ by (3.26). Since $z \cap \kappa < \kappa$ by $z \in cl(q)$, it follows that $z \cap \kappa = y \cap \kappa$. Thus, by (3.22),

$$|z \cap \kappa| \leq |z| \leq |y| = |y \cap \kappa| = |z \cap \kappa|.$$

Hence again $x \notin S_0$. — (Claim 10.1)

Clearly $q \leq_{\mathcal{CL}(S_0)} p_{\alpha^*}, p_{\beta^*}$ by (3.24). $\square$ (Lemma 10)

Lemma 11 $\mathcal{CL}(S_0)$ is κ -strategically closed.

Proof. Let σ be the following strategy of the player Nonempty in $\mathcal{G}_\kappa^{\text{II}}(\mathcal{CL}(S_0))$:

If $\alpha + 1$ st move is Nonempty's turn, then she takes $p_{\alpha+1} \leq_{\mathcal{CL}(S_0)} p_\alpha$ such that

$$(3.27) \quad |d(p_{\alpha+1})| = |d(p_\alpha)|;$$

$$(3.28) \quad \sup(d(p_\alpha)) < \sup(d_{\alpha+1}) \text{ and} \\ d(p_\alpha) \cap \kappa < d(p_{\alpha+1}) \cap \kappa.$$

Since κ is inaccessible, we can see easily by a slight modification of the proof of Lemma 9 that such a move is always possible.

Now for limit $\delta < \kappa$ suppose that $\langle p_\alpha : \alpha < \delta \rangle$ is a game played so far. Then Nonempty's move at δ according to σ is to be chosen as follows: first let $y \in \mathcal{P}_\kappa \lambda$ be such that

$$(3.29) \quad y \cap \kappa \in \kappa;$$

$$(3.30) \quad \overline{\bigcup_{\alpha < \delta} d(p_\alpha)} \subsetneq y;$$

$$(3.31) \quad |y| = |y \cap \kappa|;$$

$$(3.32) \quad \text{cf}(y \cap \kappa) = \omega \text{ and}$$

$$(3.33) \quad |y \setminus (\bigcup_{\alpha < \delta} d(p_\alpha) \cup \kappa)| = |y \cap \kappa \setminus \bigcup_{\alpha < \delta} d(p_\alpha)|.$$

In (3.30), $\bar{x}$ for $x \in \mathcal{P}_\kappa \lambda$ denotes the closure of x with respect to limit of ordinals. Note that for any $x \in \mathcal{P}_\kappa \lambda$ we have $\bar{x} \in \mathcal{P}_\kappa \lambda$.

Let $\langle \alpha_n : n \in \omega \rangle$ be an increasing sequence of ordinals cofinal in $y \cap \kappa$ (there is such a sequence by (3.32)) and let $q : y \times y \rightarrow y \cap \kappa$ be such that

$$(3.34) \quad \bigcup_{\alpha < \delta} p_\alpha \subseteq q \text{ and}$$

$$(3.35) \quad q(\alpha, n) = \alpha_n \text{ for all } \alpha \in y \setminus \bigcup_{\alpha < \delta} d(p_\alpha) \text{ and} \\ n \in \omega.$$

To see that σ is a winning strategy for the player Nonempty in $\mathcal{G}_\kappa^{\text{II}}(\mathcal{CL}(S_0))$, it is enough to show the following:

(*) If $\langle p_\alpha : \alpha < \delta \rangle$ is a game in $\mathcal{G}_\kappa^{\text{II}}(\mathcal{CL}(S_0))$ in which the player Nonempty has always played according to σ , the mapping q constructed as above by (3.29) ~ (3.35) with $y = d(q)$ is an element of $\mathcal{CL}(S_0)$.

To prove (*), We have to show that $cl(q) \cap S_0 = \emptyset$.

Suppose first that $x \in cl(q)$. If $x \subseteq d(p_\alpha)$ for some $\alpha < \delta$. Then $x \in cl(p_\alpha)$ by (3.34) and hence $x \notin S_0$.

Next, suppose that $x \not\subseteq d(p_\alpha)$ for all $\alpha < \delta$ but $x \subseteq \bigcup_{\alpha < \delta} d(p_\alpha)$. Toward a contradiction, assume that $x \in S_0$. By (3.27) and (3.31), we have

$$(3.36) \quad |(\bigcup_{\alpha < \delta} d(p_\alpha)) \cap \kappa| = |\bigcup_{\alpha < \delta} d(p_\alpha)|.$$

On the other hand, by the assumption of $x \in S_0$ and Lemma 8, we have $|x| > |x \cap \kappa|$. Hence there is some $\delta_0 < \delta$ such that

$$(3.37) \quad x \cap \kappa \leq d(p_{\delta_0}) \cap \kappa.$$

Let $\nu = \text{cf}(\delta)$ and $\mu = \text{cf}(x)$. Note that we have $\omega \leq x \cap \kappa < \text{cf}(x) = \mu$.

Let $\langle \alpha_\xi : \xi < \mu \rangle$ be a continuously increasing sequence of ordinals cofinal in $\sup(x)$. By the assumption of $x \in S_0$, there is a stationary $S^* \subseteq \mu$ such that $\{\alpha_\xi : \xi \in S^*\} \subseteq x$. For each $\xi \in S^*$, let $\beta_\xi < \delta$ be such that

$$\alpha_\xi \in d(p_{\beta_\xi}) \setminus \bigcup \{d(p_\alpha) : \alpha < \beta_\xi\}.$$

Claim 11.1 $\nu = \mu$.

⊢ Suppose that $\mu < \nu$. Then there is $\delta_1 < \delta$ such that $\delta_0 \leq \delta_1$ and $\beta_\xi < \delta_1$ for all $\xi \in S^*$. Let $x' = (x \cap \kappa) \cup \{\alpha_\xi : \xi \in S^*\}$. Then $x' \in cl(p_{\delta_1})$ and $x' \in S_0$. This is a contradiction to $p_{\delta_1} \in \mathcal{Cl}(S_0)$.

Suppose now that $\nu < \mu$. Then there is a $\delta_1 < \delta$ and a stationary $S^{**} \subseteq S^*$ such that $\delta_0 < \delta_1$ and $\beta_\xi < \delta_1$ for all $\xi \in S^{**}$. Let $x' = (x \cap \kappa) \cup \{\alpha_\xi : \xi \in S^{**}\}$. Then $x' \in cl(p_{\delta_1})$ and $x' \in S_0$. Again, this is a contradiction to $p_{\delta_1} \in \mathcal{Cl}(S_0)$. $\dashv$ (Claim 11.1)

Let $\langle \gamma_\xi : \xi < \mu \rangle$ be a continuously increasing sequence of ordinals cofinal in δ .

Claim 11.2 $\{\xi \in S^* : \beta_\xi < \gamma_\xi\}$ is non-stationary.

⊢ Otherwise, by Fodor's lemma, there is a stationary $S^{**} \subseteq S^*$ and $\beta^* < \mu$ such that $\beta_\xi < \gamma_{\beta^*}$ for all $\xi \in S^{**}$. Let $\delta_1 = \max\{\delta_0, \gamma_{\beta^*}\}$ and $x' = x \cap \kappa \cup \{\alpha_\xi : \xi \in S^{**}\}$. Then $x' \in cl(p_{\delta_1})$ and $x' \in S_0$. This is a contradiction to $p_{\delta_1} \in \mathcal{Cl}(S_0)$. $\dashv$ (Claim 11.2)

By Claim 11.2 we may assume that

$$(3.38) \quad \beta_\xi \geq \gamma_\xi \text{ for all } \xi \in S^*.$$

Let $\xi_0 \in S^* \setminus (\delta_0 + 1)$ be such that

$$(3.39) \quad \xi_0 \text{ is the limit of } S^* \cap \xi_0 \text{ and}$$

$$(3.40) \quad \beta_\xi < \gamma_{\xi_0} \text{ for all } \xi \in S^* \cap \xi_0.$$

Then by (3.30) $\beta_{\xi_0} \leq \gamma_{\xi_0}$. By (3.38), it follows that $\beta_{\xi_0} = \gamma_{\xi_0}$. Since γ_{ξ_0} is a limit, α_{ξ_0} should have been taken as a new element in Nonempty's move at the limit step. Hence by (3.35) we have $x \cap \kappa \geq d(p_{\gamma_{\xi_0}}) \cap \kappa$. But this is a contradiction to (3.37).

Finally, suppose that $x \subsetneq \bigcup_{\alpha < \delta} d(p_\alpha)$. Then by (3.35) and (3.31) it follows that $|x \cap \kappa| = |x|$. Hence again we have $x \notin S_0$. $\square$ (Lemma 11)

Lemma 12 Suppose that G is a $(V, \mathcal{Cl}(S_0))$ -generic set. In $V[G]$, κ and λ are preserved and $\mathcal{P}_\kappa \lambda = \mathcal{P}_\kappa \lambda^V$. $C = \{x \in \mathcal{P}_\kappa \lambda : x \in cl(p) \text{ for some } p \in G\}$ is a club set in $\mathcal{P}_\kappa \lambda$ disjoint from S_0 .

Proof. λ is preserved by Lemma 10 and $\kappa^+ \leq \lambda$. $\mathcal{Cl}(S_0)$ does not add new sets of size $< \kappa$ by Lemma 11. In particular κ is preserved and $\mathcal{P}_\kappa \lambda = \mathcal{P}_\kappa \lambda^V$.

By the last condition in the definition (3.11) of $\mathcal{Cl}(S_0)$, $C \cap S_1 = \emptyset$. Let $P_G = \bigcup G$. Then P_G is a

mapping by the compatibility of elements of G and the definition (3.12) of the ordering $\leq_{\mathcal{Cl}(S_0)}$. By Lemma 9 and genericity of G we have $P_G : \lambda \times \lambda \rightarrow \kappa$ and

$$C = \{x \in \mathcal{P}_\kappa \lambda : \omega \leq x \cap \kappa < \kappa, x \text{ is closed under } P_G\}.$$

Hence C is closed unbounded. $\square$ (Lemma 12)

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