

Conservative Extension of Elementary Notions in Hyting Valued Universe II

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In this paper we discuss two kinds of extensions of the Hyting valued model $V^{(\Omega)}$. The first one is the Boolean extension $V^{(B)}$ given in Part I. Here we prove that $V^{(B)}$ is a conservative extension of $V^{(\Omega)}$ for elementary notions of complex analysis. The second one is an extension of $V^{\mathcal{Q}(X)}$ to $V^{\mathcal{Q}(X \times \mathbb{C})}$, where $\mathcal{Q}(X)$ is the cHa of open sets of a topological space X and $\mathbb{C}$ is the complex number field. A complex valued continuous function on an open set of a complex number field in $V^{\mathcal{Q}(X)}$ is a complex number in $V^{\mathcal{Q}(X \times \mathbb{C})}$. By using this fact, we reformulate Oka's theorem in intuitionistic complex analysis in one variable.

Introduction

Every cHa Ω can be extended to a cBa B , and hence the model $V^{(\Omega)}$ of intuitionistic set theory has an extension $V^{(B)}$ which is a model of classical set theory. In Part I, we showed that $V^{(B)}$ is a conservative extension of $V^{(\Omega)}$ for elementary notions of real analysis. Here, in § 6 and § 7, we discuss the above for complex analysis, and then, in § 8, we consider another embedding of $V^{(\Omega)}$: The cHa of open sets of topological space X is denoted by $\mathcal{Q}(X)$. If Ω is $\mathcal{Q}(X)$ and Ω' is $\mathcal{Q}(X \times \mathbb{C})$, where $\mathbb{C}$ is the complex number field, then the natural embedding $\Omega \rightarrow \Omega'$ is extended to the embedding $V^{(\Omega)} \rightarrow V^{(\Omega')}$ and a complex valued continuous function on an open set of the complex number field in $V^{(\Omega)}$ corresponds to a complex number in $V^{(\Omega')}$. By using this fact, Rousseau reformulated and proved the Weierstrass theorems for complex analysis in several variables, in intuitionistic complex analysis in one variable. We apply the extension $V^{(\Omega)} \rightarrow V^{(B)}$ to eliminate induction process from the proof of the Weierstrass theorems in $V^{(\Omega)}$, and then reformulate Oka's theorem in intuitionistic complex analysis in one variable in $V^{(\Omega)}$ using the above extension $V^{(\Omega)} \rightarrow V^{(\Omega')}$.

6. Intuitionistic complex analysis.

In this section we assume that Ω is a spatial cHa, i.e. the set $\mathcal{Q}(X)$ of open sets of a topological space X , and show that the extension $V^{(\Omega)} \rightarrow V^{(B)}$ is conservative for notions of

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complex analysis.

Definition¹. The complex number field $\mathbb{C}^\Omega$ is an apartness field, where $x + iy \neq x' + iy'$ iff $x \neq x' \vee y \neq y'$. A complete apartness subfield of $\mathbb{C}^\Omega$ containing $\check{\mathbb{R}} + i\check{\mathbb{R}}$ is called a *complex like field*.

Now we fix a complex like field C in $V^{(\Omega)}$. We assume $\mathcal{D}(C) = \{x + iy \mid x, y \in \mathcal{D}(\mathbb{R}^\Omega)\}$ without any loss of generality. The notions of neighbourhood, open set, continuity and differentiability are defined in the same way as in § 4, i.e.

$$U_\varepsilon(a, C) = \{z \in C \mid |z - a| < \varepsilon\}, \quad \overline{U_\varepsilon(a, C)} = \{z \in C \mid |z - a| \leq \varepsilon\}.$$

$A \subseteq C$ is *open* if $\forall z \in A \exists \varepsilon (U_\varepsilon(z) \subseteq A)$, where $U_\varepsilon(z) = U_\varepsilon(z, C)$. $U_\varepsilon(a) \subset\subset A$ denotes $\exists \varepsilon' > \varepsilon (U_{\varepsilon'}(a) \subseteq A)$.

$f: \overline{U_\varepsilon(a)} \rightarrow C$ is *continuous* if $\forall \varepsilon \exists \delta \forall z, z' \in \overline{U_\varepsilon(a)} (|z - z'| < \delta \rightarrow |f(z) - f(z')| < \varepsilon)$.

$f: A \rightarrow C$ is *continuous* if $\forall U_r(a) \subset\subset A$ (f is continuous on $\overline{U_r(a)}$).

For functions $f: A \rightarrow C$ and $g: A \rightarrow C$, g is the *derivative* of f if

$$\forall U_r(a) \subset\subset A \forall \varepsilon \exists \delta \forall z, z' \in \overline{U_r(a)} (|z - z'| < \delta \rightarrow |f(z) - f(z') - g(z')(z - z')| < \varepsilon |z - z'|)$$

A function $f: A \rightarrow C$ with an open domain A is *holomorphic* or *differentiable* if f has a derivative. $A^\dagger \subseteq \mathbb{C}^B$ and $f^\dagger \subseteq \mathbb{C}^B \times \mathbb{C}^B$ are defined in the same way as in § 4.

6.1. Proposition

- (1) $\llbracket a \in C \rrbracket \leq \llbracket U_\varepsilon(a, C) = U_\varepsilon^B(a, C) \wedge \overline{U_\varepsilon(a, C)} = \overline{U_\varepsilon(a, C)^B} \rrbracket^B$
- (2) $\llbracket A \subseteq C \text{ open} \wedge u \in C \rrbracket \leq (\llbracket u \in A \rrbracket \leftrightarrow \llbracket u \in A \rrbracket^B)$
- (3) $\llbracket A \subseteq C \text{ open} \rrbracket \leq \llbracket A \subseteq C \text{ open} \wedge A^\dagger \subseteq \mathbb{C}^B \text{ open} \rrbracket^B$
- (4) $\llbracket A \subseteq C \text{ open} \wedge f: A \rightarrow C \text{ continuous} \rrbracket \leq \llbracket f: A \rightarrow C \text{ continuous} \rrbracket^B$
- (5) $\llbracket A \subseteq C \text{ open} \wedge f: A \rightarrow C \wedge g = f' \rrbracket \leq \llbracket g^\dagger = (f^\dagger)^\dagger \rrbracket^B$
- (6) $\llbracket A \subseteq C \text{ open} \wedge f, g: A \rightarrow C \text{ continuous} \rrbracket \leq \llbracket (f + g)^\dagger = f^\dagger + g^\dagger \wedge (fg)^\dagger = f^\dagger g^\dagger \rrbracket^B$
- (7) $\llbracket A \subseteq C \text{ open} \wedge f: A \rightarrow C \text{ continuous} \wedge \forall U_r(a) \subset\subset A \exists \varepsilon \forall z \in \overline{U_r(a)} (|f(z)| > \varepsilon) \rrbracket$
 $\leq \llbracket \forall z \in A^\dagger (f^\dagger(z) \neq 0) \wedge (1/f)^\dagger = 1/f^\dagger \rrbracket^B$
- (8) $\llbracket \{a_n\} \text{ sequence in } C \rrbracket \leq \llbracket \{a_n\} \text{ sequence in } C \rrbracket^B$
- (9) $\llbracket \{a_n\} \text{ Cauchy sequence in } C \rrbracket \leq \llbracket \{a_n\} \text{ Cauchy sequence in } C \wedge$
 $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty}^B a_n \rrbracket^B$
- (10) $\llbracket A \subseteq C \text{ open} \rrbracket \leq (\llbracket U_\varepsilon(a) \subset\subset A \rrbracket \leftrightarrow \llbracket U_\varepsilon(a) \subset\subset A \rrbracket^B).$

Proof. (1)-(9) are proved in the same way as those for real analysis.

(10) $\llbracket U_\varepsilon(a) \subset\subset A \rrbracket \leq \llbracket U_\varepsilon(a) \subset\subset A \rrbracket^B$ is obvious. Let $x \in \llbracket A \subseteq R \text{ open} \rrbracket \wedge \llbracket U_\varepsilon(a) \subset\subset A \rrbracket^B$. Since Ω is spatial, there exists $\varepsilon' > \varepsilon$ and $r \in \mathbb{C}$ such that $x \in \llbracket U_{\varepsilon'}(a) \subseteq A \wedge |a - \check{r}| < \frac{\varepsilon' - \varepsilon}{3} \rrbracket^B$. Let $K = \{t \in \mathbb{C} \mid |r - t| \leq \frac{\varepsilon' + \varepsilon}{2}\}$. For each $t \in K$, $x \in \llbracket \check{t} \in A \rrbracket \leq \bigvee_\delta \llbracket U_\delta(t) \subseteq A \rrbracket$. Since K is compact, there exists a finite set $\{(t_i, \delta_i) \mid i = 1, \dots, m\}$ such that $\delta_i < \frac{\varepsilon' - \varepsilon}{3}$ and $K \subseteq \bigcup_{i=1}^m \{t \in \mathbb{C} \mid |t - t_i| < \delta_i/3\}$ and

$x \in \bigwedge_{i=1}^m \llbracket U_{\delta_i}(t_i) \subseteq A \rrbracket$. Hence we have $x \in \llbracket U_\varepsilon(a) \subset \bigcup_i U_{\delta_i}(\check{t}_i) \subseteq A \rrbracket$.

Let $[\check{a}, \check{b}]_{\check{\mathbb{R}}}$ denote $\{t \in \check{\mathbb{R}} \mid \check{a} \leq t \leq \check{b}\}$ in $V^{(\alpha)}$ for $a, b \in \mathbb{R}$. Then it is obvious that $[\check{a}, \check{b}]_{\check{\mathbb{R}}} = [\check{a}, \check{b}]_{\check{\mathbb{R}}}^B$ in $V^{(B)}$. We sometimes omit the subscript $\check{\mathbb{R}}$ of $[\check{a}, \check{b}]_{\check{\mathbb{R}}}$.

Definition¹. A function $\gamma : [a, b]_{\check{\mathbb{R}}} \rightarrow C$, where $a, b \in \check{\mathbb{R}}$, is *continuously differentiable*, if γ is differentiable and γ' is continuous; and *piecewise continuously differentiable* if $\exists n \in \check{\omega} \exists t_0, \dots, t_n \in [a, b]_{\check{\mathbb{R}}} (\check{a} = t_0 < \dots < t_n = \check{b} \wedge \forall k < n (\gamma \text{ is continuously differentiable on } [t_k, t_{k+1}]_{\check{\mathbb{R}}})$. $\gamma : [a, b]_{\check{\mathbb{R}}} \rightarrow C$ is called a *path* if γ is continuous and piecewise continuously differentiable, and a path $\gamma : [a, b]_{\check{\mathbb{R}}} \rightarrow C$ is *closed* if $\gamma(a) = \gamma(b)$. The range of a path γ is denoted by $R(\gamma)$. If $\gamma : [a, b]_{\check{\mathbb{R}}} \rightarrow C$ is a continuously differentiable path and $f : R(\gamma) \rightarrow C$ is uniformly continuous, i.e.

$$\forall \varepsilon \exists N \in \check{\omega} \forall n > N \forall t_1, t_2 \in [a, b]_{\check{\mathbb{R}}} (|t_1 - t_2| < \frac{b-a}{n} \rightarrow |f(\gamma(t_1)) - f(\gamma(t_2))| < \varepsilon),$$

then $\int_{\gamma} f dz$ is defined by

$$\int_{\gamma} f dz \stackrel{\text{def}}{=} \lim_{n \rightarrow \infty} \sum_{j=1}^n f(\gamma(t_j)) \cdot \gamma'(t_j) \frac{b-a}{n},$$

where $t_j = a + \frac{j}{n}(b-a)$ and γ' is the derivative of γ . If γ is a path then $\int_{\gamma} f dz$ is the sum of the integrals of f on each piece on which γ is continuously differentiable.

It is easy to see that the extension $V^{(\alpha)} \rightarrow V^{(B)}$ is conservative for these notions, that is :

6.2. Proposition

- (1) $\llbracket \gamma : [\check{a}, \check{b}] \rightarrow C \rrbracket \leq \llbracket \gamma = \gamma^{\dagger} \rrbracket^B$
- (2) $\llbracket \gamma \text{ continuously differentiable path} \rrbracket \leq \llbracket \gamma \text{ continuously differentiable path} \rrbracket^B$
- (3) $\llbracket \gamma : [\check{a}, \check{b}] \rightarrow C \text{ path} \rrbracket \leq \llbracket \gamma : [\check{a}, \check{b}] \rightarrow C^B \text{ path} \rrbracket^B$
- (4) $\llbracket \gamma : [\check{a}, \check{b}] \rightarrow C \text{ closed path} \rrbracket \leq \llbracket \gamma : [\check{a}, \check{b}] \rightarrow C^B \text{ closed path} \rrbracket^B$
- (5) Let $p = \llbracket \gamma : [\check{a}, \check{b}] \rightarrow C \text{ continuously differentiable path} \wedge f : R(\gamma) \rightarrow C \text{ uniformly continuous} \rrbracket$. Then $p \leq \llbracket f = f^{\dagger} \wedge f : R(\gamma) \rightarrow C^B \text{ uniformly continuous} \wedge \int_{\gamma} f dz = \int_{\gamma} f dz \rrbracket^B$.

Definition¹. If $A \subseteq C$ is open, then a path $\gamma : [\check{a}, \check{b}] \rightarrow C$ is a *path in A* if

$$\exists \varepsilon \forall z \in R(\gamma) [U_\varepsilon(z) \subset C].$$

6.3. Proposition Let $p = \llbracket A \subseteq C \text{ open} \wedge \gamma : [\check{a}, \check{b}] \rightarrow C \text{ path} \rrbracket$. Then

$$p \leq (\llbracket \gamma \text{ path in } A \rrbracket \leftrightarrow \llbracket \gamma \text{ path in } A \rrbracket^B).$$

Proof. The direction $(\rightarrow)$ is proved straightforward from 6.2. We now prove $(\leftarrow)$.

$$p \wedge \llbracket \gamma \text{ path in } A \rrbracket^B \leq \llbracket \gamma \text{ continuous} \rrbracket \wedge \llbracket \exists \varepsilon \forall z \in R(\gamma) (U_\varepsilon(z) \subset A) \rrbracket^B.$$

Hence, if we set

$$p_{\varepsilon, n} = \llbracket \forall t, t' \in [\check{a}, \check{b}] (|t - t'| < \frac{\check{b} - \check{a}}{n} \rightarrow |\gamma(t) - \gamma(t')| < \varepsilon/2) \rrbracket \wedge \llbracket \forall j \leq n (U_\varepsilon(\gamma(t_j)) \subset A) \rrbracket^B,$$

where $t_j = \check{a} + \frac{j}{n}(\check{b} - \check{a})$, then $p \wedge \llbracket \gamma \text{ path in } A \rrbracket^B \leq \bigvee_{\varepsilon, n} p_{\varepsilon, n}$ and

$p_{\varepsilon, n} \leq \llbracket \forall j \leq n (U_\varepsilon(\gamma(t'_j)) \subset \subset A) \rrbracket$ by using Proposition 6.1 (10). It follows that

$$p_{\varepsilon, n} \leq \llbracket \exists \varepsilon' \forall z \in R(\gamma)(U_{\varepsilon'}(z) \subset \subset A) \rrbracket.$$

Definition¹. For $z_1, z_2 \in C$, let $[z_1, z_2]$ be the path $[\check{0}, \check{1}] \rightarrow C$ defined by $[z_1, z_2](t) = (1-t)z_1 + tz_2$. If $z_1, z_2, z_3 \in C$, then $\Delta(z_1, z_2, z_3)$ denotes the closed path consisting of $[z_1, z_2]$, $[z_2, z_3]$, $[z_3, z_1]$. If $z_1, z_2, z_3 \in C$ and $\gamma = \Delta(z_1, z_2, z_3)$, then we define

$$\begin{aligned} \text{span}(\gamma) &\stackrel{\text{def}}{=} \{a_1 z_1 + a_2 z_2 + a_3 z_3 \mid a_1, a_2, a_3 \in \check{\mathbb{R}}^+, a_1 + a_2 + a_3 = 1\} \\ \text{span}(\gamma) \subset \subset A &\stackrel{\text{def}}{\leftrightarrow} \exists \varepsilon \forall z \in \text{span}(\gamma)(U_\varepsilon(z) \subset \subset A), \end{aligned}$$

where $\check{\mathbb{R}}^+$ is the set of positive real numbers in $\check{\mathbb{R}}$.

6.4. Proposition

- (1) $\llbracket z_1, z_2 \in C \rrbracket \leq \llbracket [z_1, z_2] = [z_1, z_2]^B \rrbracket^B$
- (2) $\llbracket z_1, z_2, z_3 \in C \rrbracket \leq \llbracket \Delta(z_1, z_2, z_3) = \Delta^B(z_1, z_2, z_3) \rrbracket^B$
- (3) $\llbracket z_1, z_2, z_3 \in C \wedge \gamma = \Delta(z_1, z_2, z_3) \rrbracket \leq \llbracket \text{span}(\gamma) = \text{span}^B(\gamma) \rrbracket^B$
- (4) Let $p = \llbracket A \subseteq C \text{ open} \wedge z_1, z_2, z_3 \in C \wedge \gamma = \Delta(z_1, z_2, z_3) \rrbracket$. Then

$$p \leq (\llbracket \text{span}(\gamma) \subset \subset A \rrbracket \leftrightarrow \llbracket \text{span}(\gamma) \subset \subset A \rrbracket^B).$$

Proof. (1) $\llbracket z \in [z_1, z_2] \rrbracket^B = \bigvee_{t \in [0, 1]} \llbracket z = (\check{1} - \check{t})z_1 + \check{t}z_2 \rrbracket^B = \llbracket z \in [z_1, z_2]^B \rrbracket^B$. (2), (3) are similar to (1). (4) is proved in the same way as Proposition 6.3 by using Proposition 6.1, since for any $\varepsilon > 0$,

$$p \leq \llbracket \exists n \in \check{\omega} \exists t_1, \dots, t_n \in \text{span}(\gamma)(\text{span}(\gamma) \subseteq \bigcup U_\varepsilon(t_i)) \rrbracket.$$

Definition¹. Let A be an open set in C and a function $f: A \rightarrow C$ be continuous. Then f is *analytic on A* if for each $\gamma = \Delta(z_1, z_2, z_3)$ with $z_1, z_2, z_3 \in A$, $\text{span}(\gamma) \subset \subset A$ implies $\int_\gamma f dz = 0$.

6.5. Proposition $\llbracket A \subseteq C \wedge f: A \rightarrow C \text{ analytic on } A \rrbracket \leq \llbracket f^\dagger \text{ analytic on } A^\dagger \rrbracket^B$.

Proof. Let $p = \llbracket A \subseteq C \text{ open} \wedge f: A \rightarrow C \text{ analytic on } A \rrbracket$, then

$$p \leq \llbracket A^\dagger \subseteq \mathbb{C}^B \text{ open} \wedge f^\dagger: A^\dagger \rightarrow \mathbb{C}^B \text{ continuous} \wedge A \text{ dense in } A^\dagger \rrbracket^B.$$

Hence, by using Proposition 6.4(4),

$$\begin{aligned} p &\leq \llbracket \forall \gamma = \Delta(z_1, z_2, z_3)(\text{span}(\gamma) \subset \subset A \rightarrow \int_\gamma f dz = 0) \wedge A \text{ dense in } A^\dagger \rrbracket^B \\ &\leq \llbracket \forall \gamma = \Delta(z_1, z_2, z_3)(\text{span}(\gamma) \subset \subset A^\dagger \rightarrow \int_\gamma f dz = 0) \rrbracket^B. \end{aligned}$$

Definition. An open set A of C is *convex* if for each $z_1, z_2 \in A$, $[z_1, z_2]$ is a path in A .

6.6. Proposition $\llbracket A \subseteq C \text{ convex} \rrbracket \leq \llbracket A^\dagger \subseteq \mathbb{C}^B \text{ convex} \rrbracket^B$.

Proof. Let $p = \llbracket A \subseteq C \text{ convex} \rrbracket \wedge \llbracket z_1, z_2 \in A^\dagger \rrbracket^B$. Then

$$p \leq \bigvee_{y_i, \varepsilon} \llbracket U_\varepsilon(z_i) \subset \subset A^\dagger \wedge |y_i - z_i| \in \varepsilon/3 \rrbracket^B \wedge \llbracket y_i \in A \rrbracket \text{ for } i = 1, 2. \text{ By Proposition 6.1(10)}$$

$$p \leq \bigvee_{y_i, \varepsilon} \llbracket U_{\varepsilon/3}(z_i) \subseteq U_{2\varepsilon/3}(y_i) \rrbracket^B \wedge \llbracket U_{2\varepsilon/3}(y_i) \subset\subset A \rrbracket \quad \text{for } i=1, 2.$$

It follows that

$$p \leq \bigvee_{y_1, y_2, \varepsilon} \llbracket U_{\varepsilon/3}(tz_1 + (1-t)z_2) \subseteq U_{2\varepsilon/3}(t\check{y}_1 + (1-t)\check{y}_2) \subset\subset A^\dagger \rrbracket^B.$$

Definition¹. Let $A \subseteq C$ be open. Paths $\gamma_i: [0, 1] \rightarrow A$, $i=0, 1$, are *homotopic in A* if there exists a function $\sigma: [0, 1]^2 \rightarrow A$ satisfying the following conditions:

- (a) σ is continuous and piecewise differentiable in each component,
- (b) $\sigma(\check{0}, t) = \gamma_0(t)$, $\sigma(\check{1}, t) = \gamma_1(t)$ and $\exists \varepsilon \forall t, t' \in [\check{0}, \check{1}] (U_\varepsilon(\sigma(t, t')) \subset\subset A)$.

6.7. Proposition

$$\llbracket A \subseteq C \text{ open} \wedge \gamma_0, \gamma_1 \text{ homotopic paths in } A \rrbracket \leq \llbracket \gamma_0, \gamma_1 \text{ homotopic in } A \rrbracket^B.$$

Proof. By Propositions 6.1, 6.2.

Definition¹. An open subset A of C is *connected* if for any $z_1, z_2 \in A$ there exists a path $\gamma: [\check{0}, \check{1}] \rightarrow A$ in A such that $\gamma(0) = z_1$ and $\gamma(1) = z_2$. A closed path γ in A is *homotopic to 0* if γ is homotopic to a constant path.

6.8. Proposition

- (1) $\llbracket A \subseteq C \text{ open and connected} \rrbracket \leq \llbracket A \text{ connected} \rrbracket^B$
- (2) $\llbracket A \subseteq C \text{ open} \wedge \gamma \text{ closed path in } A \text{ and homotopic to } 0 \rrbracket \leq \llbracket \gamma \text{ homotopic to } 0 \rrbracket^B.$

Proof. By Propositions 6.2, 6.3, 6.7.

Definition¹. $e^z \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \frac{z^n}{n!}$ for $z \in C$.

6.9. Proposition $\llbracket z \in C \rrbracket \leq \llbracket e^z = (e^z)^B \rrbracket^B.$

Proof. By Proposition 6.1.

Definition¹. For $A, A' \subseteq C$,

$$\begin{aligned} A - A' &\stackrel{\text{def}}{=} \{z \in A \mid \neg z \in A'\} \\ A \dot{-} A' &\stackrel{\text{def}}{=} \{z \in A \mid \exists \varepsilon \forall z' \in A' (|z - z'| > \varepsilon)\} \end{aligned}$$

6.10. Proposition

- (1) $\llbracket A, A' \subseteq C \text{ open} \rrbracket \leq \llbracket (A \dot{-} A') \subseteq (A \dot{-} A')^B \rrbracket^B$
- (2) $\llbracket A \subseteq C \text{ open} \wedge z \in C \rrbracket \leq \llbracket (A \dot{-} \{z\}) = (A \dot{-} \{z\})^B \rrbracket^B,$
- (3) $\llbracket A \subseteq C \text{ open} \wedge \gamma \text{ path in } A \rrbracket \leq \llbracket (A \dot{-} R(\gamma))^\dagger = (A^\dagger \dot{-} R(\gamma))^B \rrbracket^B.$

Proof. (1) is proved straightforward. (2) follows from Proposition 6.1 (2) and 3.2 .

(3) follows from the fact that if $A \subseteq C$ is open and γ is a path in A then $A \dot{-} R(\gamma)$ is open.

Definition¹. If $z_0 \in C$ and γ is a closed path in $C \dot{-} \{z_0\}$, then

$$O(\gamma, z_0) \stackrel{\text{def}}{=} \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z - z_0}.$$

$O(\gamma, z_0)$ is an integer and called the *index* of z_0 with respect to γ .

6.11. Proposition $\llbracket z_0 \in C \wedge \gamma \text{ closed path in } C \dot{-} \{z_0\} \rrbracket \leq \llbracket O(\gamma, z_0) = O^B(\gamma, z_0) \rrbracket^B$.

Proof. By Proposition 6.2.

Definition¹. For $r \in \check{\mathbb{Q}}^+$ (i.e. r is a positive rational number) and $z_0 \in C$, the path $\gamma : [\check{0}, \check{1}] \rightarrow C$ given by $\gamma(t) = z_0 + re^{2\pi it}$ is denoted by $C(z_0, r)$.

Then obviously we have:

6.12. Proposition $\llbracket r \in \check{\mathbb{Q}}^+ \wedge z_0 \in C \rrbracket \leq \llbracket C(z_0, r) = C^B(z_0, r) \rrbracket^B$.

7. Application II.

Here we apply the results in §6 to prove theorems of complex analysis in $V^{(\Omega)}$, where Ω is a spatial cHa. Let C be a complex like field in V^Ω .

7.1. Theorem¹ If f, g are holomorphic functions $A \rightarrow C$ on an open subset A of C , then

$$(1) (f + g)' = f' + g', (fg)' = f'g + fg'$$

$$(2) \forall U_r(a) \subset \subset A \exists \varepsilon \forall z \in \overline{U_r(a)} (|f(z)| > \varepsilon) \rightarrow ((1/f)' = -f'/f^2).$$

Proof. (1) Let $p = \llbracket A \subseteq C \text{ open} \wedge f, g : A \rightarrow C \text{ holomorphic} \rrbracket$. Then

$$\begin{aligned} p &\leq I \llbracket A \subseteq C \text{ open} \wedge f, g : A \rightarrow C \text{ holomorphic} \rrbracket^B \\ &\leq I \llbracket (f + g)'^\dagger = (f' + g')^\dagger \rrbracket^B \leq \llbracket (f + g)' = f' + g' \rrbracket, \end{aligned}$$

by using Proposition 6.1 (5) and (6). Similarly, $p \leq \llbracket (fg)' = f'g + fg' \rrbracket$. (2) is proved in the same way as (1) by using Proposition 6.1 (5) and (7).

7.2. Theorem¹ If $A \subseteq C$ is open and $f : A \rightarrow C$ is holomorphic, then f is analytic.

Proof. Set $p = \llbracket A \subseteq C \text{ open} \wedge f : A \rightarrow C \text{ holomorphic} \wedge \gamma = \Delta(z_1, z_2, z_2) \wedge \text{span}(\gamma) \subset \subset A \rrbracket$. Then

$$\begin{aligned} p &\leq \llbracket A^\dagger \subseteq \mathbb{C}^B \text{ open} \wedge f^\dagger : A^\dagger \rightarrow \mathbb{C}^B \text{ holomorphic} \wedge \gamma = \Delta(z_1, z_2, z_3) \wedge \text{span}(\gamma) \subset \subset A \rrbracket^B \\ &\leq \llbracket \int_\gamma f^\dagger dz = 0 \rrbracket^B. \end{aligned}$$

$$\text{Hence } p \leq \llbracket \int_\gamma f dz = 0 \rrbracket.$$

The following Theorems 7.4–7.10 can be proved similarly.

7.4. Theorem¹ If $A \subseteq C$ is open, $f : A \rightarrow C$ is analytic and γ_0, γ_1 are homotopic paths in A , then $\int_{\gamma_0} f dz = \int_{\gamma_1} f dz$.

7.5. Theorem¹ If $A \subseteq C$ is open, $z_0 \in A$, $\gamma : [\check{a}, \check{b}] \rightarrow A$ is a path in $A \dot{-} \{z_0\}$, and $\gamma(\check{a}) = z_1$ and $\gamma(\check{b}) = z_2$, then $\exp\left(\int_\gamma \frac{dz}{z - z_0}\right) = \frac{z_2 - z_0}{z_1 - z_0}$.

7.6. Theorem^I (*Cauchy's integral formula for a disk*) If $A \subseteq C$ is open, $f: A \rightarrow C$ is holomorphic and $U_r(z_0) \subset\subset A$, then for any $z \in U_r(z_0)$,

$$f(z) = \frac{1}{2\pi i} \int_{C(z_0, r)} \frac{f(\xi)}{\xi - z} d\xi.$$

7.7. Theorem^I If γ is a closed path in C , $h: R(\gamma) \rightarrow C$ is uniformly continuous and $n \in \mathfrak{W}$, then the function

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{h(\xi)}{(\xi - z)^n} d\xi$$

is differentiable on $C \setminus R(\gamma)$ and its derivative is

$$g(z) = \frac{n}{2\pi i} \int_{\gamma} \frac{h(\xi)}{(\xi - z)^{n+1}} d\xi.$$

7.8. Theorem^I (*Cauchy's integral formula*) If $A \subseteq C$ is open, $f: A \rightarrow C$ is holomorphic, $z_0 \in A$ and γ is a closed path in $A \setminus \{z_0\}$ which is homotopic to 0 in A , then

$$O(\gamma, z_0)f(z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - z_0} dz.$$

7.9. Theorem^I (*Cauchy's inequality*) If $f: U_r(z_0) \rightarrow C$ is holomorphic and bounded by M , then for any $n \in \mathfrak{W}$

$$|f^{(n)}(z_0)| \leq n! M / r^n.$$

7.10. Theorem^I (*Liouville's Theorem*) If $f: C \rightarrow C$ is holomorphic and bounded by M , then f is constant.

7.11. Theorem^I (*Maximum principle*) If $A \subseteq C$ is open, $f: A \rightarrow C$ is holomorphic and $U_r(a) \subset\subset A$, then

$$|f(z)| \leq M \text{ on } C(a, r) \text{ implies } |f(z)| \leq M \text{ on } U_r(a).$$

Proof. Let

$$p = \llbracket A \subseteq C \text{ open} \wedge f: A \rightarrow C \text{ holomorphic} \wedge U_r(a) \subset\subset A \wedge (|f(z)| \leq M \text{ on } C(a, r)) \wedge z' \in \overline{U_r(a)} \rrbracket.$$

Then by using the Maximum principle in classical analysis in $V^{(B)}$, $p \leq \llbracket |f(z')| \leq M \rrbracket^B$.

Hence $p \leq \llbracket |f(z')| \leq M \rrbracket$.

8. Complex analysis in several variables.

C. Rousseau reformulated and proved the Weierstrass division theorem and preparation theorem for complex analysis in several variables, in intuitionistic complex analysis in one variable. (cf. [8]). In §8.1 we show that the induction process in the proof of the

intuitionistic Weierstrass theorems can be eliminated by using the extension $V^{(\alpha)} \rightarrow V^{(B)}$. In § 8.2 we reformulate the Oka's theorem in intuitionistic complex analysis in one variable.

8.1. Weierstrass theorems.

Let Ω be a spatial cHa and C be a complex like field in $V^{(\alpha)}$.

8.1.1. Theorem¹ Let $A \subseteq C$ be an open neighbourhood of 0 and $f: A \rightarrow C$ be holomorphic. If $f^{(k)}(0) \neq 0$, then either $\exists i < k (f^{(i)}(0) \neq 0)$ or there exists $r > 0$ such that $U_r(0) \subset\subset A$ and the following condition is satisfied: For any holomorphic and bounded $g: U_r(0) \rightarrow C$ there exists a unique holomorphic $h: U^r(0) \rightarrow C$ and a unique polynomial P of degree $< k$ such that $g = fh + P$.

We refer to the above condition as $D(r, k)$.

Proof. Rousseau [8].

8.1.2. Theorem¹ (Weierstrass division theorem) Let $f: A \rightarrow C$ be holomorphic, where A is a neighbourhood of 0 in C . If $f^{(k)}(0) \neq 0$, then there exist $i \leq k$ and $r > 0$ such that $U_r(0) \subset\subset A$ and $D(r, i)$.

Proof. Set $p = \llbracket A \subseteq C \text{ open} \wedge 0 \in A \wedge f: A \rightarrow C \text{ holomorphic} \wedge f^{(k)}(0) \neq 0 \rrbracket$. Then $p \leq \bigvee_{i \leq k} \llbracket \forall j < i (f^{(j)}(0) = 0) \wedge f^{(i)}(0) \neq 0 \rrbracket^B$. Since $\llbracket f^{(i)}(0) \neq 0 \rrbracket^B = \llbracket f^{(i)}(0) \neq 0 \rrbracket$, we have

$$p \wedge \llbracket f^{(i)}(0) \neq 0 \rrbracket^B \leq \llbracket \exists j < i (f^{(j)}(0) \neq 0) \vee \exists r > 0 (U_r(0) \subset\subset A \wedge D(r, i)) \rrbracket$$

by Theorem¹ 8.1.1. Further,

$$p \wedge \llbracket \forall j < i (f^{(j)}(0) = 0) \rrbracket^B \wedge \llbracket \exists j < i (f^{(j)}(0) \neq 0) \rrbracket = 0.$$

Therefore,

$$p \leq \bigvee_{i \leq k} \llbracket \exists r > 0 (U_r(0) \subset\subset A \wedge D(r, i)) \rrbracket.$$

8.1.3. Theorem¹ (Weierstrass preparation theorem) Let $f: A \rightarrow C$ be holomorphic, where A is a neighbourhood of 0. If $f^{(k)}(0) \neq 0$, then there exist $i \leq k$, $W \subseteq A$ neighbourhood of 0, a unique P monic polynomial of degree i and a unique $h: W \rightarrow C$ holomorphic and invertible such that $f = hP$.

Proof. Set $p = \llbracket A \subseteq C \text{ open} \wedge 0 \in A \wedge f: A \rightarrow C \text{ holomorphic} \wedge f^{(k)}(0) \neq 0 \rrbracket$ and $p_i = p \wedge \llbracket \forall j < i (f^{(j)}(0) = 0) \wedge f^{(i)}(0) \neq 0 \rrbracket^B$. Then

$$p_i \leq \llbracket \exists r > 0 (U_r(0) \subset\subset A \wedge D(r, i)) \rrbracket.$$

as we saw in the proof of 8.1.2. In $V^{(\alpha)}$, if $\exists r > 0 (U_r(0) \subset\subset A \wedge D(r, i))$ then there exists a unique $h: U_r(0) \rightarrow C$ holomorphic, and a unique polynomial P of degree $< i$ such that $z^i = fh + P$. Taking the i th derivative of both sides,

$$i! = \sum_{j=0}^i \binom{i}{j} f^{(j)}(0) h^{(i-j)}(0) \neq 0.$$

Hence, $p_i = p_i \wedge \llbracket \sum_{j=0}^i \binom{i}{j} f^{(j)}(0) h^{(i-j)}(0) \neq 0 \rrbracket \leq \llbracket h(0) \neq 0 \rrbracket$. Therefore,

$$p \leq \bigvee_{i \leq k} p_i \leq \bigvee_{i \leq k} \bigvee_{r' > 0} \llbracket U_{r'}(0) \subset \subset A \wedge \exists h : U_{r'}(0) \rightarrow C \text{ holomorphic and invertible } \exists P \text{ polynomial of } \deg < i \text{ such that } f = h^{-1}(z^i - P) \rrbracket.$$

The uniqueness, $p_i \wedge \llbracket f = hP = h'P' \wedge \deg P = \deg P' = i \rrbracket \leq \llbracket h = h' \wedge P = P' \rrbracket$, follows from the uniqueness in Theorem¹ 8.1.1.

Let $\Omega_n = \mathcal{O}(\mathbb{C}^n)$, $n=1, 2, \dots$. Then Ω_{n-1} is embedded in Ω_n by $p \rightarrow p \times \mathbb{C}$, and $V^{(\Omega_{n-1})}$ is embedded in $V^{(\Omega_n)}$. The set $\mathbb{C}^{\Omega_n}$ of complex numbers in $V^{(\Omega_n)}$ represents the sheaf of germs of complex valued continuous functions on $\mathbb{C}^n$. It is known that if A is an open set of a complex like field and f is a holomorphic function on A in $V^{(\Omega_{n-1})}$, then A represents an open set $\bar{A}$ of $\mathbb{C}^n$ and f a complex valued continuous function $\bar{f}$ on $\bar{A}$ which is holomorphic in the last variable. Hence $\bar{f}$ is represented by a complex number $\hat{f}$ in $V^{(\Omega_n)}(\text{cf. [8]})$. Let H_n be the set of complex numbers in $V^{(\Omega_n)}$ which represents the sheaf of germs of holomorphic function on $\mathbb{C}^n$, i.e.

$$\begin{cases} \mathcal{D}(H_n) = \{z \in \mathcal{D}(\mathbb{C}^{\Omega_n}) \mid \bar{z} \text{ is holomorphic}\} \\ H_n(z) = \llbracket z \in \mathbb{C}^{\Omega_n} \rrbracket. \end{cases}$$

It is known that H_n is a complex like field in $V^{(\Omega_n)}$ and we have

8.1.4. Lemma $\llbracket A \subseteq H_{n-1} \text{ open} \wedge f : A \rightarrow H_{n-1} \text{ holomorphic} \rrbracket = 1$ iff $\llbracket \hat{f} \in H_n \rrbracket = \bar{A}$.
(cf. [8])

The notion of coherent sheaf is translated as the following notion of c -finite in $V^{(\Omega)}$.

Definition¹. Let R be a ring. R -module M is c -finite if:

- (a) M is finitely generated, and
- (b) for $f_1, \dots, f_n \in M$,

$$\mathcal{R}(f_1, \dots, f_n) = \{ \langle g_1, \dots, g_s \rangle \in R^s \mid \sum_{i=1}^s g_i f_i = 0 \}$$

is finitely generated.

A complex like field C is c -finite if C is c -finite as a C -module.

Then the Oka's theorem asserts that H_n is c -finite on $V^{(\Omega_n)}$, and is reformulated as a theorem in $V^{(\Omega)}$ as follows.

8.1.5. Theorem¹ (Oka's theorem in $V^{(\Omega)}$) Let C be a complex like field and c -finite.

If $f_1, \dots, f_s$ are holomorphic on a neighbourhood of 0 and $f_j^{(n_j)}(0) \neq 0, j=1, \dots, s$,

then there exists a neighbourhood W of 0 such that

$$\mathcal{R}(f_1, \dots, f_s) = \{ \langle g_1, \dots, g_s \rangle \mid \forall i \leq s (g_i : W \rightarrow C \text{ holomorphic}) \wedge \sum_{i=1}^s g_i f_i = 0 \text{ on } W \}$$

is finitely generated.

Proof. Since the Weierstrass theorem 8.1.2, 8.1.3 can be generalized for a finite set of holomorphic functions instead of one holomorphic function f , there exists a neighbourhood W of 0 , $i_j \leq n_j$, h_j , P_j ($j=1, \dots, s$) such that for each j

- (1) $f_j = h_j P_j$ on W
- (2) h_j and $1/h_j$ are holomorphic on W
- (3) P_j is a monic polynomial of degree i_j
- (4) if $g : W \rightarrow C$ is holomorphic then there exists a unique holomorphic $g_1 : W \rightarrow C$ and a unique polynomial g_2 of degree $< i_j$ such that $g = g_1 P_j + g_2$.

Hence it suffices to prove the theorem for the case $f_j = P_j$ ($j=1, \dots, s$). Further, we may assume $m = i_s \geq i_1, \dots, i_{s-1}$ without any loss of generality.

Each s -tuples of the form

$$(\underbrace{0, \dots, 0}_{i-1}, P_s, \underbrace{0, \dots, 0}_{s-j-1}, -P_j) \quad (\ast)$$

is in $\mathcal{R} = \mathcal{R}(f_1, \dots, f_s)$. Let $\mathcal{R}^*$ be the subset of $\mathcal{R}$ consisting of s -tuples of polynomials of degree $\leq m-1$. Then $\mathcal{R}^*$ is finitely generated since C is c -finite. Thus, it suffices to show that $\mathcal{R}$ is generated by $\mathcal{R}^*$ and the set of s -tuples of the form $(\ast)$. Let $(a_1, \dots, a_s) \in \mathcal{R}$. Dividing each a_j by P_s ,

$$a_j = q_j P_s + u_j, \quad j = 1, \dots, s-1,$$

where u_j is a polynomial of degree $\leq m-1$. Since

$$\sum_{j=1}^s a_j P_j = \left(\sum_{j=1}^{s-1} q_j P_j + a_s \right) P_s + \sum_{j=1}^{s-1} u_j P_j = 0,$$

we have $\langle u_1, \dots, u_s \rangle \in \mathcal{R}$ for $u_s = \sum_{j=1}^{s-1} q_j P_j + a_s$. $-\sum_{j=1}^{s-1} u_j P_j$ is of degree $\leq 2m-1$

and equal to $u_s P_s$. It follows that u_s is of degree $\leq m-1$, and $\langle u_1, \dots, u_s \rangle \in \mathcal{R}^*$.

References.

- 1) Bishop, E. : *Foundations of Constructive Analysis*. McGraw Hill 1967.
- 2) Fourman, M. and Scott, D.S. : *Sheaves and logic. Lecture Notes in Mathematics* 753(1977), Springer-Verlag, pp. 302-401.
- 3) Funayama, N. : On imbedding infinitely distributive lattice completely isomorphically into Boolean algebras, *Nagoya Math. J.* 15 (1959), pp. 71-81.
- 4) Grayson, R.J. : Heyting valued models for intuitionistic set theory. *Lecture Notes in Mathematics* 753 (1977), Springer-Verlag, pp. 402-414.
- 5) Gunning, R. and Rossi, H. : *Analytic Functions of Several Complex Variables*. Prentice-Hall, 1965.
- 6) Hormander, L. : *An Introduction to Complex Analysis in Several Variables*. North-Holland, 1973.

- 7) Rasiowa, H. and Sikorski, R. : *The Mathematics of Metamathematics*. Warsaw, 1963.
- 8) Rousseau, C. : Topos theory and complex analysis. *Lecture Notes in Mathematics* 753 (1977), Springer-Verlag, pp. 623-659.
- 9) Takeuti, G. and Titani, S. : Heyting valued universes of intuitionistic set theory. *Lecture Notes in Mathematics* 891 (1980), Springer-Verlag, pp. 189-306.