

Conservative Extension of Elementary Notions in Hyting Valued Universe I

Satoko TITANI* and Gaisi TAKEUTI**

(Received April 20, 1988)

It is known that every complete Heyting algebra (cHa) Ω can be extended to a complete Boolean algebra (cBa) B . It follows that an Ω -valued model $V^{(\Omega)}$ of intuitionistic set theory can be extended to a Boolean valued model $V^{(B)}$ of classical set theory. We show in this paper that the Boolean valued model $V^{(B)}$ is a conservative extension of a Heyting valued model $V^{(\Omega)}$ for elementary notions of analysis. This enables us to prove theorems in intuitionistic analysis by reducing them to corresponding theorems in classical analysis.

Introduction.

Funayama showed in [3] that every complete Hyting algebra (cHa) is embedded in a complete Boolean algebra (cBa). Later Rasiowa and Sikorski obtained another embedding of cHa in cBa. For a cHa Ω let $V^{(\Omega)}$ be the Ω -valued universe of intuitionistic set theory given in Grayson [4]. An embedding of Ω in a cBa B , $\Omega \rightarrow B$, is extended to an embedding $V^{(\Omega)} \rightarrow V^{(B)}$, and $V^{(B)}$ is a model of ZF set theory in classical logic. We show in this paper that $V^{(B)}$ is a conservative extension of $V^{(\Omega)}$ for elementary notions of real analysis such as real numbers, derivatives, integrals, etc. This makes many delicate intuitionistic arguments unnecessary for development of analysis in $V^{(\Omega)}$. Then we prove theorems in intuitionistic real analysis in $V^{(\Omega)}$ by reducing them to corresponding theorems in classical analysis in $V^{(B)}$. The intuitionistic complex analysis will be discussed in part II.

1. Preliminaries.

Here we review the Hyting valued model $V^{(\Omega)}$ of intuitionistic set theory in Grayson [4] and Takeuti-Titani [9].

Let Ω be a complete Heyting algebra (cHa). For each ordinal α , $V_\alpha^{(\Omega)}$ is defined inductively as follows: $V_0^{(\Omega)} = \emptyset$; $V_{\alpha+1}^{(\Omega)}$ is the set of all Ω -valued functions on subsets of $V_\alpha^{(\Omega)}$, and $V_\alpha^{(\Omega)} = \bigcup_{\beta < \alpha} V_\beta^{(\Omega)}$ for a limit ordinal α . The Ω -valued universe $V^{(\Omega)}$ is then $V^{(\Omega)} = \bigcup_\alpha V_\alpha^{(\Omega)}$. We denote the domain of $u \in V^{(\Omega)}$ by $\mathcal{D}(u)$.

The truth value $\llbracket \varphi \rrbracket$ of a set theoretical formula φ on $V^{(\Omega)}$ is defined in the usual way

*Natural Sciences-I

**Department of Mathematics, University of Illinois

from:

$$\begin{aligned}\llbracket u = v \rrbracket &= \bigwedge_{x \in \mathcal{D}(u)} (u(x) \rightarrow \llbracket x \in v \rrbracket) \wedge \bigwedge_{y \in \mathcal{D}(v)} (v(y) \rightarrow \llbracket y \in u \rrbracket), \\ \llbracket u \in v \rrbracket &= \bigvee_{y \in \mathcal{D}(v)} (\llbracket u = y \rrbracket \wedge v(y)).\end{aligned}$$

1.1 Lemma For $u \in V^{(\Omega)}$ and a formula $\varphi(x)$,

$$\begin{aligned}\llbracket \forall x \in u \varphi(x) \rrbracket &= \bigwedge_{x \in \mathcal{D}(u)} (u(x) \rightarrow \llbracket \varphi(x) \rrbracket), \\ \llbracket \exists x \in u \varphi(x) \rrbracket &= \bigvee_{x \in \mathcal{D}(u)} (u(x) \wedge \llbracket \varphi(x) \rrbracket).\end{aligned}$$

The universe V is embedded into $V^{(\Omega)}$ by the mapping $a \rightarrow \check{a}$ where $\check{a}$ is defined by $\mathcal{D}(\check{a}) = \{\check{t} \mid t \in a\}$ and $\check{a}(\check{t}) = 1$.

1.2 Lemma For $a, b \in V$,

$$\llbracket \check{a} = \check{b} \rrbracket = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{if } a \neq b \end{cases}, \quad \llbracket \check{a} \in \check{b} \rrbracket = \begin{cases} 1 & \text{if } a \in b \\ 0 & \text{if } a \notin b \end{cases}.$$

Definition. For $u \in V^{(\Omega)}$ and $p \in \Omega$, we define the *restriction of u on p* , $u \upharpoonright p$, by $\mathcal{D}(u \upharpoonright p) = \mathcal{D}(u)$ and $(u \upharpoonright p)(x) = \llbracket x \in u \rrbracket \wedge p$ for $x \in \mathcal{D}(u \upharpoonright p)$. Thus, we have:

$$p \leq \llbracket u = u \upharpoonright p \rrbracket \text{ and } \llbracket x \in u \upharpoonright p \rrbracket = \llbracket x \in u \rrbracket \wedge p.$$

$v \in V^{(\Omega)}$ is said to be *definite* if for each $x \in \mathcal{D}(u)$, $u(x) = \llbracket x \in u \rrbracket$.

1.3 Lemma For each $u \in V^{(\Omega)}$ there exists a definite $u' \in V^{(\Omega)}$ such that $\llbracket u = u' \rrbracket = 1$.

Proof. Let $\mathcal{D}(u') = \mathcal{D}(u)$ and $u'(x) = \llbracket x \in u \rrbracket$ for $x \in \mathcal{D}(u')$.

Definiton. $u \in V^{(\Omega)}$ is a *sheaf representation* (SR) if

- (a) u is definite,
- (b) for every $v \in V^{(\Omega)}$ there exists $v' \in \mathcal{D}(u)$ such that $\llbracket v \in u \rrbracket \leq \llbracket v = v' \rrbracket$, and
- (c) for every $p \in \Omega$ and $v \in \mathcal{D}(u)$, there exists $v' \in \mathcal{D}(u)$ such that $\llbracket v \upharpoonright p = v' \rrbracket = 1$.

1.4 Lemma For every $u \in V^{(\Omega)}$ there exists a SR $u' \in V^{(\Omega)}$ such that $\llbracket u = u' \rrbracket = 1$. (cf. [9]).

1.5 Lemma If X is definite, Y is SR and $\llbracket h: X \rightarrow Y \rrbracket = p$, then there exists a function $\tilde{h}: \mathcal{D}(X) \rightarrow \mathcal{D}(Y)$ such that

- (a) $p \wedge \llbracket x \in X \rrbracket \leq \llbracket \tilde{h}(x) \in Y \wedge \langle x, \tilde{h}(x) \rangle \in h \rrbracket$ for $x \in \mathcal{D}(X)$,
- (b) $p \wedge \llbracket x = y \in X \rrbracket \leq \llbracket \tilde{h}(x) = \tilde{h}(y) \rrbracket$ for $x, y \in \mathcal{D}(X)$,

where $h: X \rightarrow Y$ stands for $: h \subseteq X \times Y \wedge \forall x \in X \exists ! y \in Y (\langle x, y \rangle \in h)$. Further if $h' \in V^{(\Omega)}$ is defined by $\mathcal{D}(h') = \{\langle x, \tilde{h}(x) \rangle \mid x \in \mathcal{D}(X)\}$ and $h'(x, \tilde{h}(x)) = p \wedge \llbracket x \in X \rrbracket$,

then $p \leq \llbracket h = h' \rrbracket$. (cf [9])

Notations. The set of natural numbers is denoted by ω , and rational, real, complex number fields by $\mathbb{Q}, \mathbb{R}, \mathbb{C}$, respectively. The set of natural numbers in $V^{(\omega)}$ coincides with $\check{\omega}$ and hence the rational number field in $V^{(\omega)}$ is $\check{\mathbb{Q}}$. However, this is not true for real and complex number fields. Real and complex number fields in $V^{(\omega)}$ are denoted by $\mathbb{R}^\omega$ and $\mathbb{C}^\omega$, respectively. ε, δ are used to denote a positive rational number in both of V and $V^{(\omega)}$. ‘Theorem’ and ‘Definition’ in the intuitionistic set theory on $V^{(\omega)}$ are written with the superscript I as Theorem^I and Definition^I.

2. Extension $V^{(\omega)} \rightarrow V^{(B)}$.

Let Ω be a cHa. Ω can be embedded in a cBa. That is, there exists an embedding $e: \Omega \rightarrow B$ preserving meet and infinite join: $e(a \wedge b) = e(a) \wedge e(b)$ and $e(\bigvee_i a_i) = \bigvee_i^B e(a_i)$ for $a, b, a_i \in \Omega$, where $\bigvee^B$ denotes the infinite join in B . We omit the superscript B from $\bigvee^B$ if there is no possible confusion. Identifying $a \in \Omega$ with $e(a)$, let $\Omega \subseteq B$. For $p \in B$ we define the *interior* of p , $I(p) = \bigvee \{q \in \Omega \mid q \leq p\}$. Then we have:

2.1 Proposition

- $I(p) \leq p$ for $p \in B$; $I(p) = p$ for $p \in \Omega$.
- (2) $I(\bigwedge_j^B p_j) = \bigwedge_j I(p_j)$ for $\{p_j\} \subseteq B$, where $\bigwedge_j^B$ denotes the infinite meet in B .
- $(p \rightarrow q) = I(p \xrightarrow{B} q)$ for $p, q \in \Omega$, where $(p \xrightarrow{B} q) = (-p \vee q)$ in B .

Now construct the Ω -valued and B -valued universes $V^{(\omega)}$ and $V^{(B)}$ defined in § 1. Since Ω is a subalgebra of B , $V^{(\omega)}$ is contained in $V^{(B)}$ that is a model of ZF set theory in classical logic. We denote the valuation in $V^{(B)}$ by $\llbracket \cdot \rrbracket^B$ to distinguish from the valuation $\llbracket \cdot \rrbracket$ in $V^{(\omega)}$. If a set $\varphi(u_1, \dots, u_n)$ is defined from constants $u_1, \dots, u_n$ in $V^{(\omega)}$ then the corresponding set defined in $V^{(B)}$ is denoted by $\varphi^B(u_1, \dots, u_n)$ or $(\varphi(u_1, \dots, u_n))^B$, e.g. $\bigcup^B u$ is the set in $V^{(B)}$ such that $\llbracket x \in \bigcup^B u \leftrightarrow \exists v \in u (x \in v) \rrbracket^B = 1$.

By using Proposition 2.1 we can prove:

2.2 Proposition If $u, v \in V^{(\omega)}$, then

- (a) $\llbracket u = v \rrbracket \leq \llbracket u = v \rrbracket^B$, $\llbracket u \in v \rrbracket \leq \llbracket u \in v \rrbracket^B$,
- (b) $\llbracket \{u, v\} = \{u, v\}^B \wedge \langle u, v \rangle = \langle u, v \rangle^B \rrbracket^B = 1$.

2.3 Proposition If A is a set and $h, X, Y \in V^{(\omega)}$, then

$$\llbracket X \subseteq \check{A} \wedge h: X \rightarrow Y \rrbracket \leq \llbracket h: X \rightarrow Y \rrbracket^B.$$

Proof. Let $p = \llbracket X \subseteq \check{A} \wedge h: X \rightarrow Y \rrbracket$. We may assume that $\mathcal{D}(X) \subseteq \mathcal{D}(\check{A})$, X is definite and Y is SR. By Lemma 1.5, we may also assume that there exists a function $\tilde{h}: \mathcal{D}(X) \rightarrow \mathcal{D}(Y)$ such that $\mathcal{D}(h) = \{\langle x, \tilde{h}(x) \rangle \mid x \in \mathcal{D}(X)\}$ and $h \langle x, \tilde{h}(x) \rangle = p \wedge \llbracket x \in X \rrbracket$ for $x \in \mathcal{D}(X)$. It follows that $p \leq \llbracket h: X \rightarrow Y \rrbracket^B$.

3. Real and complex number fields.

Definition¹. Real number is defined as a Dedekind cut of $\check{\mathbb{Q}}$, that is, u is a *real* if there exist $L, U \subseteq \check{\mathbb{Q}}$ such that $u = \langle L, U \rangle$ and the following conditions are satisfied.

- R1. $\exists r, s \in \check{\mathbb{Q}} (r \in L \wedge s \in U)$
- R2. $\forall r \in \check{\mathbb{Q}} \neg (r \in L \wedge r \in U)$
- R3. $\forall r \in \check{\mathbb{Q}} (r \in L \leftrightarrow \exists s \in \check{\mathbb{Q}} (r < s \wedge s \in L))$
- R4. $\forall r \in \check{\mathbb{Q}} (r \in U \leftrightarrow \exists s \in \check{\mathbb{Q}} (s < r \wedge s \in U))$
- R5. $\forall r \in \check{\mathbb{Q}} (s < r \rightarrow s \in L \vee r \in U).$

If $\llbracket u \text{ is a real} \rrbracket = p$, then we define $L_u, U_u \in V^{(\Omega)}$ by $\mathcal{D}(L_u) = \mathcal{D}(U_u) = \mathcal{D}(\check{\mathbb{Q}})$,
 $L_u(\check{r}) = \llbracket \exists L, U \subseteq \check{\mathbb{Q}} (u = \langle L, U \rangle \wedge \check{r} \in L) \rrbracket \wedge p$ and $U_u(\check{r}) = \llbracket \exists L, U \subseteq \check{\mathbb{Q}} (u = \langle L, U \rangle \wedge \check{r} \in U) \rrbracket \wedge p$. Then $p \leq \llbracket u = \langle L_u, U_u \rangle \rrbracket$. Define $\mathbb{R}^\Omega \in V^{(\Omega)}$ by

$$\mathcal{D}(\mathbb{R}^\Omega) = \{ \langle L_u, U_u \rangle \mid \llbracket u \text{ is a real} \rrbracket \neq 0 \}, \text{ and } \mathbb{R}^\Omega(L_u, U_u) = \llbracket u \text{ is a real} \rrbracket.$$

$\mathbb{R}^\Omega$ is SR and $\llbracket \mathbb{R}^\Omega = \{ u \in \mathcal{P}(\check{\mathbb{Q}}) \times \mathcal{P}(\check{\mathbb{Q}}) \mid u \text{ is a real} \} \rrbracket = 1$. Let $\mathbb{R}^B = \{ u \in \mathcal{P}(\check{\mathbb{Q}}) \times \mathcal{P}(\check{\mathbb{Q}}) \mid u \text{ is a real} \}^B$. Then we have:

3.1 Proposition

(1) If $\langle L, U \rangle \in \mathcal{D}(\mathbb{R}^\Omega)$ and $r \in \check{\mathbb{Q}}$, then

$$\llbracket \check{r} \in L \rrbracket = L(\check{r}) = \llbracket \check{r} \in L \rrbracket^B \text{ and } \llbracket \check{r} \in U \rrbracket = U(\check{r}) = \llbracket \check{r} \in U \rrbracket^B.$$

(2) $\llbracket \mathbb{R}^\Omega \subseteq \mathbb{R}^B \rrbracket^B = 1$.

Proof. (1) follows from the fact that $\llbracket \check{r} = \check{r}' \rrbracket \neq 0$ iff $r = r'$. (2) By using Proposition 2.1, 2.2 and (1), we can see that $\llbracket Ri \rrbracket \leq \llbracket Ri \rrbracket^B$ for each condition Ri , $i=1, \dots, 5$, of real numbers.

Definition¹. For real numbers $u = \langle L, U \rangle$ and $u' = \langle L', U' \rangle$,

$$\begin{aligned} u < u' &\stackrel{\text{def}}{\leftrightarrow} \exists r (r \in L' \cap U); \quad u \leq u' \stackrel{\text{def}}{\leftrightarrow} L \subseteq L' \wedge U' \subseteq U; \\ u \neq u' &\stackrel{\text{def}}{\leftrightarrow} u < u' \vee u' < u. \end{aligned}$$

We can easily prove the following propositions.

3.2 Proposition For $u, u' \in \mathcal{D}(\mathbb{R}^\Omega)$,

$$\llbracket u < u' \rrbracket = \llbracket u < u' \rrbracket^B \text{ and } \llbracket u = u' \rrbracket = I \llbracket u = u' \rrbracket^B.$$

3.3 Proposition If $p = \bigwedge_{i=1,2} \llbracket f_i : X \rightarrow \mathbb{R}^\Omega \rrbracket \leq \bigwedge_{i=1,2} \llbracket f_i : X \rightarrow \mathbb{R}^\Omega \rrbracket^B$, then

$$p \leq (\llbracket f_1 = f_2 \rrbracket \leftrightarrow I \llbracket f_1 = f_2 \rrbracket^B).$$

The set of rational numbers $\check{\mathbb{Q}}$ is considered as a subset of $\mathbb{R}^\Omega$ by identifying $r \in \check{\mathbb{Q}}$ with the pair $\langle \{r' \in \check{\mathbb{Q}} \mid r' < r\}, \{r' \in \check{\mathbb{Q}} \mid r' > r\} \rangle$, i.e. $\mathcal{D}(L_r) = \mathcal{D}(U_r) = \mathcal{D}(\check{\mathbb{Q}})$ and $L_r(\check{t}) = \llbracket \check{t} < r \rrbracket$, $U_r(\check{t}) = \llbracket \check{t} > r \rrbracket$ for $t \in \check{\mathbb{Q}}$.

Definition¹. Let r, s, u be variables ranging over $\check{\mathbb{Q}}$, and $u = \langle L_u, U_u \rangle$, $v = \langle L_v, U_v \rangle$ be reals. Then operations on $\mathbb{R}^\Omega$ are defined as follows.

- (1) $u + v \stackrel{\text{def}}{=} \langle L, U \rangle$ where $L = \bigcup_{r < u, s < v} L_{r+s}$ and $U = \bigcup_{r > u, s > v} U_{r+s}$.
- (2) $-u \stackrel{\text{def}}{=} \langle L, U \rangle$ where $L = \bigcup_{r > u} L_{-r}$ and $U = \bigcup_{r < u} U_{-r}$.
- (3) If $t \geq 0$, $tu \stackrel{\text{def}}{=} \langle L, U \rangle$ where $L = \bigcup_{r < u} L_{tr}$ and $U = \bigcup_{r > u} U_{tr}$; if $t < 0$, $tu \stackrel{\text{def}}{=} -(-t)u$.
- (4) $uv \stackrel{\text{def}}{=} \langle L, U \rangle$ where $L = \bigcup_{r < u, s < v} L_{(su+rv-rs)} \cup \bigcup_{r > u, s > v} L_{(su+rv-rs)}$ and $U = \bigcup_{r < u, s > v} U_{(su+rv-rs)} \cup \bigcup_{r > u, s < v} U_{(su+rv-rs)}$.
- (5) If $u > 0$, $u^{-1} \stackrel{\text{def}}{=} \langle L, U \rangle$ where $L = \bigcup_{r > u} L_{r^{-1}}$ and $U = \bigcup_{r < u} U_{r^{-1}}$;
if $u < 0$, $u^{-1} \stackrel{\text{def}}{=} -(-u)^{-1}$.
- (6) $\max(u, v) \stackrel{\text{def}}{=} \langle L_u \cup L_v, U_u \cap U_v \rangle$.
- (7) $|u| \stackrel{\text{def}}{=} \max(u, -u)$.

Then, Obviously, $u + v, -u, uv, u^{-1}, \max(u, v), |u|$ are reals, and $\mathbb{R}^\Omega$ is a field. Furthermore, $\mathbb{R}^\Omega$ is an apartness field, i.e. $\mathbb{R}^\Omega$ is a field satisfying the following conditions:

$$\begin{aligned} a \neq b &\rightarrow \neg a = b; \quad \neg a \neq b \rightarrow a = b; \quad a \neq b \rightarrow a \neq c \vee b \neq c; \quad 1 \neq 0; \\ a \neq b &\rightarrow a + c \neq b + c; \quad a \neq b \wedge c \neq 0 \rightarrow ac \neq bc; \quad a \neq 0 \rightarrow a^{-1}a = 1 \wedge a^{-1} \neq 0. \end{aligned}$$

3.4 Proposition For $u, v \in \mathcal{D}(\mathbb{R}^\Omega)$,

$$\begin{aligned} \llbracket u + v = (u + v)^B \rrbracket &= 1; \quad \llbracket -u = (-u)^B \rrbracket^B = 1; \quad \llbracket uv = (uv)^B \rrbracket^B = 1; \\ \llbracket u \neq 0 \rrbracket &\leq \llbracket u^{-1} = (u^{-1})^B \rrbracket^B; \quad \llbracket \max(u, v) = \max^B(u, v) \rrbracket^B = 1; \quad \llbracket |u| = |u|^B \rrbracket = 1. \end{aligned}$$

Proof. We prove only the first equation, since the others can be proved similarly. It is obvious that $L_u(\check{r}) = \llbracket \check{r} < u \rrbracket$ and $U_u(r) = \llbracket \check{r} > u \rrbracket$ for $r \in \mathbb{Q}$. Hence,

$$L_{u+v}(\check{r}) = \llbracket \exists s \in L_u \exists t \in L_v (\check{r} \in L_{s+t}) \rrbracket = \llbracket \exists s \in L_u \exists t \in L_v (\check{r} \in L_{s+t}) \rrbracket^B = L_{(u+v)^B}(\check{r}).$$

Similarly, $U_{u+v}(\check{r}) = U_{(u+v)^B}(\check{r})$ and hence $\llbracket u + v = (u + v)^B \rrbracket^B = 1$.

The complex number field $\mathbb{C}^\Omega$ in $V^{(\Omega)}$ is defined by $\mathcal{D}(\mathbb{C}^\Omega) = \{u + iv \mid u, v \in \mathcal{D}(\mathbb{R}^\Omega)\}$ and $\mathbb{C}^\Omega(u + iv) = \llbracket u \in \mathbb{R}^\Omega \wedge v \in \mathbb{R}^\Omega \rrbracket$ introducing i such that $\llbracket i^2 = -1 \rrbracket = 1$. It is obvious that $\llbracket \mathbb{C}^\Omega = \{u + iv \mid u, v \in \mathbb{R}^\Omega\} \rrbracket = 1$ and $\mathbb{C}^\Omega$ is a SR. In $V^{(\Omega)}$, absolute value of a complex number is defined as usual, $|u + iv| = \sqrt{u^2 + v^2}$, and apartness relation on $\mathbb{C}^\Omega$ is defined by $a \neq b \stackrel{\text{def}}{\leftrightarrow} |a - b| \neq 0$. $\mathbb{C}^\Omega$ is an apartness field and satisfies $u + iv \neq u' + iv' \leftrightarrow u \neq u' \vee v \neq v'$.

3.5 Proposition

- (1) If $u, v \in \mathcal{D}(\mathbb{C}^\Omega)$, then $\llbracket u = v \rrbracket = I \llbracket u = v \rrbracket^B$
- (2) If $p = \bigwedge_{i=1,2} \llbracket f_i : A \rightarrow \mathbb{C}^\Omega \rrbracket \leq \bigwedge_{i=1,2} \llbracket f_i : A \rightarrow \mathbb{C}^\Omega \rrbracket^B$, then $p \leq (\llbracket f_1 = f_1 \rrbracket \leftrightarrow I \llbracket f_1 = f_2 \rrbracket^B)$

4. Intuitionistic real analysis.

Definition¹. A sequence in a set X is defined as a function $a: \check{\omega} \rightarrow X$, and denoted by $\{a_n\}$. We use m, n to denote variables ranging over $\check{\omega}$. A sequence $\{a_n\}$ in $\mathbb{R}^\Omega$ converges to b ($\lim_{n \rightarrow \infty} a_n = b$) if $\forall \varepsilon \exists N \in \check{\omega} \forall n > N (|a_n - b| < \varepsilon)$. A sequence $\{a_n\}$ in $\mathbb{R}^\Omega$ is a Cauchy sequence if

$$\forall \varepsilon \exists N \in \check{\omega} \forall m, n > N (|a_m - a_n| < \varepsilon).$$

A subset X of $\mathbb{R}^\Omega$ is complete if every Cauchy sequence converges to an element of X . A complete apartness subfield of $\mathbb{R}^\Omega$ containing $\check{\mathbb{R}}$ is called a real like field. $\check{\mathbb{R}}$ is an apartness subfield of $\mathbb{R}^\Omega$, but not necessarily complete. If Ω is a complete Boolean algebra of regular open sets of $\mathbb{R}$, then $\check{\mathbb{R}}$ is not complete. If $a, b \in R \subseteq \mathbb{R}^\Omega$, then the intervals $(a, b)_R$, $[a, b]_R$ in R are defined by

$$(a, b)_R = \{x \in R \mid a < x < b\}, \quad [a, b]_R = \{x \in R \mid a \leq x \leq b\}.$$

If there is no possible confusion, we omit R from $(a, b)_R$, $[a, b]_R$. The intervals $(a - \varepsilon, a + \varepsilon)_R$, $[a - \varepsilon, a + \varepsilon]_R$ are denoted by $U_\varepsilon(a, R)$, $\overline{U_\varepsilon(a, R)}$, or simply by $U_\varepsilon(a)$, $\overline{U_\varepsilon(a)}$. A subset A of a real like field R is open if $\forall a \in A \exists \varepsilon (U_\varepsilon(a, R) \subseteq A)$. We use the notation $U_\varepsilon(a) \subset\subset A$ to denote $\exists \varepsilon' > \varepsilon (U_{\varepsilon'}(a) \subseteq A)$. It is obvious that $U_\varepsilon(a, R)$ is an open subset of a real like field R .

Now we fix a real like field R , where we assume $\mathcal{D}(R) = \mathcal{D}(\mathbb{R}^\Omega)$ without any loss of generality. Then R is a sheaf representation.

4.1 Theorem¹ If $a, b \in R$ and $a \leq b$, then the interval $[a, b]_R$ is totally bounded, i.e.

$$\forall \varepsilon \exists n \in \check{\omega} \exists a_1, \dots, a_n \in [a, b]_R \forall x \in [a, b]_R \exists i \leq n (x \in U_\varepsilon(a_i)).$$

Proof. Let $p = \llbracket a, b \in R \wedge a \leq b \rrbracket$ and $\varepsilon > 0$ be given. Then

$p \leq \bigvee_{a', b' \in \mathbb{Q}} \llbracket \check{a}' < a \leq b < \check{b}' \wedge |\check{a}' - a| < \varepsilon' \wedge |\check{b}' - b| < \varepsilon' \rrbracket$. Hence the theorem follows from the fact that $[\check{a}', \check{b}']_R$ is totally bounded.

4.2 Proposition $\llbracket a, b \in R \wedge a \leq b \rrbracket \leq \llbracket (a, b)_R = (a, b)_R^B \wedge [a, b]_R = [a, b]_R^B \rrbracket^B$.

We now see that the extension $V^{(\Omega)} \rightarrow V^{(B)}$ is conservative for the concept of open set and continuous function. In $V^{(B)}$, let

$$(a, b)_R^\dagger = (a, b)_{\mathbf{R}^B}, \quad [a, b]_R^\dagger = [a, b]_{\mathbf{R}^B} \text{ for } a, b \in R,$$

and if $A \subseteq R$ is open, let

$$A^\dagger = \bigcup \{U_\varepsilon(a)^\dagger \mid U_\varepsilon(a) \subseteq A\}.$$

4.3 Proposition For $A \in V^{(\Omega)}$ and $x \in \mathcal{D}(R)$,

$$(1) \llbracket A \subseteq R \text{ open} \rrbracket \leq (\llbracket x \in A \rrbracket \leftrightarrow \llbracket x \in A \rrbracket^B),$$

$$(2) \llbracket A \subseteq R \text{ open} \rrbracket \leq \llbracket A \subseteq R \text{ open} \wedge A = A^\dagger \cap R \rrbracket^B.$$

Proof. (1) $\llbracket x \in A \rrbracket \leq \llbracket x \in A \rrbracket^B$ by Proposition 3.1.

$$\begin{aligned} \llbracket A \subseteq R \text{ open} \rrbracket \wedge \llbracket x \in A \rrbracket^B &\leq \bigvee_{t \in \mathcal{D}(A)} \llbracket A \subseteq R \text{ open} \rrbracket \wedge \llbracket x = t \rrbracket^B \wedge A(t) \\ &\leq \bigvee_{t \in \mathcal{D}(A)} \bigvee_{\varepsilon} \llbracket U_{\varepsilon}(t) \subseteq A \wedge |x - t| < \varepsilon \rrbracket \leq \llbracket x \in A \rrbracket. \end{aligned}$$

$$(2) \llbracket A \subseteq R \text{ open} \rrbracket \wedge A(t) \leq \bigvee_{\varepsilon} \llbracket U_{\varepsilon}(t, R) \subseteq A \rrbracket \leq \bigvee_{\varepsilon} \llbracket U_{\varepsilon}^B(t, R) \subseteq A \rrbracket^B$$

for $t \in \mathcal{D}(A)$, by using (1) and Proposition 4.2. Hence we have

$\llbracket A \subseteq R \text{ open} \rrbracket \leq \llbracket A \subseteq R \text{ open} \rrbracket^B$. The other part is obvious.

Definition¹. A function $f: [a, b]_R \rightarrow R$ is *continuous* if

$\forall \varepsilon \exists \delta \forall u, v \in [a, b]_R (|u - v| < \delta \rightarrow |f(u) - f(v)| < \varepsilon)$, and if A is an open subset of R , then a function $f: A \rightarrow R$ is *continuous* if f is cotinuous on $\overline{U_r(a)}$ for every $U_r(a) \subset\subset A$.

4.4 Proposition For $a, b \in \mathcal{D}(R)$ and $A \in V^{(\Omega)}$,

- (1) $\llbracket a, b \in R \wedge a \leq b \wedge f: [a, b]_R \rightarrow R \text{ continuous} \rrbracket \leq \llbracket f: [a, b]_R \rightarrow R \text{ continuous} \rrbracket^B$
 $\leq \llbracket \exists ! g: [a, b]_R^{\dagger} \rightarrow \mathbb{R}^B \text{ continuous such that } f = g \text{ on } [a, b]_R \rrbracket^B$
- (2) $\llbracket A \subseteq R \text{ open} \wedge f: A \rightarrow R \text{ continuous} \rrbracket \leq \llbracket f: A \rightarrow R \text{ continuous} \rrbracket^B$
 $\leq \llbracket \exists ! g: A^{\dagger} \rightarrow \mathbb{R}^B \text{ continuous such that } f = g \text{ on } A \rrbracket^B$

The unique g in (1) and (2) will be denoted by $f^{\dagger}$.

Proof. (1) Set $p = \llbracket a, b \in R \wedge a \leq b \wedge f: [a, b]_R \rightarrow R \text{ continuous} \rrbracket$. By Lemma 1.3 and 1.5, we may assume that $[a, b]_R$ is definite, $\tilde{f}: \mathcal{D}[a, b]_R \rightarrow \mathcal{D}(R)$ and

$$\mathcal{D}(f) = \{ \langle x, \tilde{f}(x) \rangle \mid x \in \mathcal{D}[a, b]_R \}, \quad f \langle x, \tilde{f}(x) \rangle = p \wedge \llbracket x \in [a, b]_R \rrbracket.$$

Then for $x \in \mathcal{D}[a, b]_R$,

$$\begin{aligned} p \wedge [a, b]_R(x) &\leq \llbracket \tilde{f}(x) \in R \wedge \langle x, \tilde{f}(x) \rangle \in f \rrbracket \leq \llbracket \exists y \in R (\langle x, y \rangle \in f) \rrbracket^B \\ p \wedge [a, b]_R(x) \wedge \llbracket \langle x, y \rangle \in f \rrbracket^B &\leq \bigvee_{t \in \mathcal{D}[a, b]_R} \llbracket \langle x, y \rangle = \langle t, \tilde{f}(t) \rangle \rrbracket^B \wedge p \\ &\leq \bigvee_{t \in \mathcal{D}[a, b]_R} \llbracket |\tilde{f}(x) - \tilde{f}(t)| < \varepsilon \wedge |y - \tilde{f}(t)| < \varepsilon \rrbracket, \end{aligned}$$

by using continuity of f in $V^{(\Omega)}$. It follows that

$$p \wedge [a, b]_R(x) \leq \llbracket \langle x, y \rangle \in f \rightarrow y = \tilde{f}(x) \rrbracket^B.$$

Hence $p \leq \llbracket f: [a, b]_R \rightarrow R \rrbracket^B$. Continuity of f in $V^{(B)}$ follows from :

$$p \leq \bigwedge_{\varepsilon} \bigvee_{\sigma} \bigwedge_{x, y \in \mathcal{D}[a, b]_R} \llbracket x, y \in [a, b]_R \wedge |x - y| < \delta \rightarrow |\tilde{f}(x) - \tilde{f}(y)| < \varepsilon \rrbracket^B.$$

The second inequality follows from the fact that $[a, b]_R$ is dense in $A^{\dagger}$.

(2) is proved in the same way as (1).

Let A be an open set of R in $V^{(\Omega)}$. It is known that in $V^{(\Omega)}$ if f, g are continuous functions $A \rightarrow R$ then so are $f + g$ and fg ; and if a continuous function $f: A \rightarrow R$ satisfies $\forall U_r(a) \subset\subset A \exists \varepsilon \forall z \in \overline{U_r(a)} (|f(z)| > \varepsilon)$, then $1/f$ is continuous. Hence we have:

4.5 Proposition Let $A \subseteq R$ be open in $V^{(\Omega)}$.

- (1) $\llbracket f, g: A \rightarrow R \text{ continuous} \rrbracket \leq \llbracket f + g, fg: A \rightarrow R \text{ continuous} \wedge$
 $(f + g)^{\dagger} = f^{\dagger} + g^{\dagger} \wedge (fg)^{\dagger} = f^{\dagger} g^{\dagger} \rrbracket^B$

$$(2) \llbracket f : A \rightarrow R \text{ continuous} \wedge \forall U_r(a) \subset \subset A \exists \varepsilon \forall z \in \overline{U_r(a)} (|f(z)| > \varepsilon) \rrbracket \\ \leq \llbracket \forall z \in A^\dagger (f^\dagger(z) \neq 0) \wedge (1/f)^\dagger = 1/f^\dagger \rrbracket^B.$$

Definition¹. Let I be a closed interval in R (i.e. $a, b \in R$, $a \leq b$ and $I = [a, b]_R$), and f, g be functions $I \rightarrow R$. Then g is a *derivative of f* if

$$\forall \varepsilon \exists \delta \forall u, v \in [a, b]_R (|u - v| < \delta \rightarrow |f(u) - f(v) - g(v)(u - v)| < \varepsilon |u - v|).$$

For an open set $A \subseteq R$ and functions $f : A \rightarrow R$, $g : A \rightarrow R$, g is a *derivative of f* if g is a derivative of f on $\overline{U_\varepsilon(a)}$ for every $U_\varepsilon(a) \subset \subset A$. A function $f : A \rightarrow R$ is *differentiable* if f has a derivative. Then we have the following theorem in $V^{(\Omega)}$.

4.6 Theorem¹ Let $A \subseteq R$ be open.

- (1) If g_1 and g_2 be derivatives of $f : A \rightarrow R$, then $g_1 = g_2$. The unique derivative of f is denoted by f' .
- (2) If $g : A \rightarrow R$ is the derivative of $f : A \rightarrow R$, then g is continuous.
- (3) If $f : A \rightarrow R$ is differentiable, then f is continuous.

4.7 Proposition Let $p = \llbracket I \text{ is an interval in } R \rrbracket$, i.e.

$$p = \llbracket \exists a, b \in R (a \leq b \wedge (I = (a, b)_R \vee I = [a, b]_R)) \rrbracket.$$

Then $p \wedge \llbracket f : I \rightarrow R \rrbracket \wedge \llbracket f' : I \rightarrow R \text{ is the derivative of } f \rrbracket$

$$\leq \llbracket f' \text{ is the derivative of } f \rrbracket^B \wedge \llbracket (f')^\dagger \text{ is the derivative of } f^\dagger \rrbracket^B.$$

Proof. Similar to Proposition 4.4.

4.8 Proposition

- (1) $\llbracket \{a_n\} \text{ is a sequence in } X \rrbracket \leq \llbracket \{a_n\} \text{ is a sequence in } X \rrbracket^B$.
- (2) $\llbracket \{a_n\} \text{ is a Cauchy sequence in } R \rrbracket$
 $\leq \llbracket \{a_n\} \text{ is a Cauchy sequence in } R \text{ and } \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty}^B a_n \rrbracket^B$.

Proof. (1) follows from Proposition 2.3.

(2) Straightforward.

Definition¹. Let $a, b \in R$, $a \leq b$ and a function $f : [a, b]_R \rightarrow R$ be continuous.

Then the sequence $\{a_n\}$ with $a_n = \frac{b-a}{n} \sum_{k=0}^{n-1} f\left(\frac{(n-k)a + kb}{n}\right)$ is a Cauchy sequence, and the integral $\int_a^b f(x)dx$ is defined as $\lim_{n \rightarrow \infty} a_n$.

4.9 Proposition

$$\llbracket a, b \in R \wedge a \leq b \wedge f : [a, b]_R \rightarrow R \text{ continuous} \rrbracket \leq \llbracket \int_a^b f(x)dx = \int_a^b f^\dagger(x)dx \rrbracket^B$$

where $\int_a^b f(x)dx$, $\int_a^b f^\dagger(x)dx$ are integrals in $V^{(\Omega)}$, $V^{(B)}$, respectively.

Proof. By Proposition 3.4, 4.4, 4.8.

4.10 Proposition Let $r, s \in \mathbb{Q}$, $r < s$, $u \in V^{(B)}$ and

$$p = \llbracket a, b \in R \wedge a < b \wedge (I = [a, b]_R \vee I = (a, b)_R) \wedge f : I \rightarrow R \text{ continuous} \rrbracket.$$

Then $p \wedge \llbracket u \in I^\dagger \wedge \check{r} < f^\dagger(u) < \check{s} \rrbracket^B \leq \llbracket \exists c \in I (\check{r} < f(c) < \check{s}) \rrbracket$.

Proof.

$$\begin{aligned} & p \wedge \llbracket u \in I^\dagger \wedge \check{r} < f^\dagger(u) < \check{s} \rrbracket^B \\ & \leq \llbracket I \subseteq I^\dagger \text{ dense} \wedge f^\dagger: I^\dagger \rightarrow \mathbb{R}^B \text{ continuous} \wedge u \in I^\dagger \wedge r < f^\dagger(u) < s \rrbracket^B. \\ & \leq \llbracket \exists c \in I (\check{r} < f^\dagger(c) < \check{s}) \rrbracket^B = \llbracket \exists c \in I (\check{r} < f(c) < \check{s}) \rrbracket. \end{aligned}$$

In $V^{(\omega)}$, if I is a closed interval in R and a function $f: I \rightarrow R$ is continuous, then there exists an upper bound of f on I , since I is totally bounded (Theorem¹ 4.1).

Definition¹.

$$\|f\|_I \leq a \stackrel{\text{def}}{\leftrightarrow} (\forall x \in I)(|f(x)| \leq a)$$

$$\|f\|_I < a \stackrel{\text{def}}{\leftrightarrow} (\exists a' < a)(\|f\|_I \leq a').$$

4.11 Proposition Let $p = \llbracket a, b \in R \wedge a \leq b \wedge I = [a, b]_R \wedge f: I \rightarrow R \text{ continuous} \rrbracket$.

Then $p \leq (\llbracket \|f\|_I < \varepsilon \rrbracket \leftrightarrow \llbracket \|f\|_I < \varepsilon \rrbracket^B \leftrightarrow \llbracket \|f^\dagger\|_{I^\dagger} < \varepsilon \rrbracket^B)$.

Proof. $p \wedge \llbracket \|f\|_I < \varepsilon \rrbracket \leq \llbracket \|f\|_I < \varepsilon \rrbracket^B$ is obvious.

$$p \wedge \llbracket \|f\|_I < \varepsilon \rrbracket^B \leq \bigvee_{\varepsilon' < \varepsilon} p \wedge \llbracket \forall x \in I (|f(x)| < \varepsilon') \rrbracket^B.$$

Let $q_{\varepsilon', n} = p \wedge \llbracket \|f\|_I < \varepsilon \rrbracket^B \wedge \llbracket \forall x \in I (|f(x)| < \varepsilon') \rrbracket^B \wedge$

$$\llbracket \forall x, y \in I \left(|x - y| < \frac{b - a}{n} \rightarrow |f(x) - f(y)| < \frac{\varepsilon - \varepsilon'}{2} \right) \rrbracket$$

Since $q_{\varepsilon', n} \leq \llbracket \forall k < n (|f(\frac{(n-k)a + kb}{n})| < \varepsilon') \rrbracket$,

$$p \wedge \llbracket \|f\|_I < \varepsilon \rrbracket^B \leq \bigvee_{\varepsilon', n} q_{\varepsilon', n} \leq \bigvee_{\varepsilon'} \llbracket \forall x \in I (|f(x)| < \frac{\varepsilon + \varepsilon'}{2} < \varepsilon) \rrbracket.$$

Definition¹. Let $a, b \in R$, $a \leq b$ and $I = [a, b]_R$ or $(a, b)_R$. Let $\forall n \in \check{\omega} (f_n: I \rightarrow R)$. Then

$$f_n \rightarrow f \text{ on } I \stackrel{\text{def}}{\leftrightarrow} \lim_{n \rightarrow \infty} \|f_n - f\|_I = 0 \text{ (i.e. } \forall \varepsilon \exists N \in \check{\omega} \forall n > N (\|f_n - f\|_I < \varepsilon)).$$

4.12 Proposition Let $p = \llbracket a, b \in R \wedge a \leq b \wedge I = [a, b]_R \rrbracket$. Then

$$\begin{aligned} & p \wedge \llbracket f: I \rightarrow R \text{ continuous} \wedge \forall n \in \check{\omega} (f_n: I \rightarrow R \text{ continuous}) \wedge (f_n \rightarrow f \text{ on } I) \rrbracket \\ & \leq \llbracket f_n \rightarrow f \text{ on } I \rrbracket^B. \end{aligned}$$

Proof. Immediate from Proposition 4.11.

Definition¹. For $x \in R$,

$$e^x \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \cos x \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \quad \sin x \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

For positive $x \in R$, $\log x \stackrel{\text{def}}{=} \int_1^x \frac{dt}{t}$,

4.13 Proposition

$$(1) \llbracket x \in R \rrbracket \leq \llbracket e^x = (e^x)^B \wedge \cos x = (\cos x)^B \wedge \sin x = (\sin x)^B \rrbracket^B$$

$$(2) \llbracket x \in R \wedge x > 0 \rrbracket \leq \llbracket \log x = (\log x)^B \rrbracket^B.$$

Proof. By Propositions 3.5, 4.8, 4.9.

Definition¹. If $a, b, c, d \in R$, $a \leq b$, $c \leq d$, and a function $f: [a, b]_R \times [c, d]_R \rightarrow R$ is continuous, then the sequence $\{z_n\}$ with

$$z_n = \frac{b-a}{n} \frac{d-c}{n} \sum_{i,j=0}^{n-1} f\left(\frac{(n-i)a+ib}{n}, \frac{(n-j)c+jd}{n}\right)$$

is a Cauchy sequence and the integral $\int \int_{[a,b] \times [c,d]} f(x, y) dx dy$ is defined as $\lim_{n \rightarrow \infty} z_n$.

4.14 Proposition

$$\llbracket a, b, c, d \in R \wedge a \leq b \wedge c \leq d \wedge f: [a, b] \times [c, d] \rightarrow R \text{ continuous} \rrbracket$$

$$\leq \llbracket \int \int_{[a,b] \times [c,d]} f(x, y) dx dy = \int \int_{[a,b] \times [c,d]} f^\dagger(x, y) dx dy \rrbracket^B,$$

where $\int \int_{[a,b] \times [c,d]} f^\dagger(x, y) dx dy$ is the integral of $f^\dagger$ in $V^{(B)}$.

Proof. Similar to Proposition 4.9.

5. Application I.

Since $V^{(B)}$ is a model of ZF in classical logic, every theorem of usual analysis holds in $V^{(B)}$. Here we apply the results of § 4 to prove theorems of intuitionistic real analysis in $V^{(\Omega)}$ by reducing them to the corresponding theorems in $V^{(B)}$.

5.1 Theorem¹ (Rolle's theorem) Let I be a closed interval $[a, b]$ in R . If $f: I \rightarrow R$ is differentiable and $f(a) = f(b)$, then for any $\varepsilon > 0$ there exists $x \in I$ such that $|f'(x)| < \varepsilon$.

Proof. Let $p = \llbracket I = [a, b] \text{ interval in } R \wedge f: I \rightarrow R \text{ differentiable} \wedge f(a) = f(b) \rrbracket$.

Then

$$\begin{aligned} p &\leq \llbracket f^\dagger: I^\dagger \rightarrow \mathbb{R}^B \text{ differentiable} \wedge f^\dagger(a) = f^\dagger(b) \rrbracket^B \\ &\leq \llbracket \exists c \in I^\dagger (f^\dagger(c) = f'^\dagger(c) = 0) \rrbracket^B \\ &\leq \llbracket \exists c \in I (|f'(c)| < \varepsilon) \rrbracket \end{aligned}$$

by using Proposition 4.7, 4.10 and Rolle's theorem in $V^{(B)}$.

Similarly, we can prove the following intuitionistic theorems in $V^{(\Omega)}$.

5.2 Theorem¹ (Mean value theorem) Let I be a closed interval $[a, b]$ in R . If $f: I \rightarrow R$ is differentiable, then for any ε there exists $c \in I$ such that $|f(b) - f(a) - f'(c)(b - a)| < \varepsilon$.

5.3 Theorem¹ (Taylor's theorem) Let I be a closed interval in R , and $f: I \rightarrow R$ be $n+1$ times differentiable. Then for any $a, b \in I$ and ϵ there exists $c \in R$ such that $\min(a, b) \leq c \leq \max(a, b)$ and

$$|f(b) - \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (b-a)^k - \frac{f^{(n+1)}(c)}{n!} (b-c)^n (b-a)| < \epsilon.$$

5.4 Theorem¹ Let I be a closed interval in R and $f_n: I \rightarrow R$ be differentiable for each $n \in \check{\omega}$. If $g: I \rightarrow R$ is continuous, $f_n \rightarrow f$ on I and $f'_n \rightarrow g$ on I , then $f' = g$.

Proof. Set $p = \llbracket I \text{ closed interval in } R \wedge f_n \rightarrow f \text{ on } I \wedge \forall n \in \check{\omega} (f_n \text{ differentiable}) \wedge$

$$f'_n \rightarrow g \text{ on } I \wedge f: I \rightarrow R \text{ continuous} \rrbracket.$$

$p \leq \llbracket f_n^\dagger \rightarrow f^\dagger \text{ on } I^\dagger \wedge \forall n \in \check{\omega} (f_n \text{ differentiable}) \wedge f_n^{\dagger'} \rightarrow g^\dagger \text{ on } I^\dagger \wedge g^\dagger: I^\dagger \rightarrow \mathbb{R}^B \text{ continuous} \rrbracket^B$. Hence by using the corresponding theorem in $V^{(B)}$, we have

$$p \leq I \llbracket f^{\dagger'} = g^\dagger \rrbracket^B \leq \llbracket f' = g \rrbracket.$$

Similarly we can prove intuitionistic theorems for elementary functions which are given as equations, such as

$$\frac{d}{dx} e^x = e^x, \quad e^{x+y} = e^x e^y \text{ for } x, y \in R;$$

$$\frac{d}{dx} \log x = \frac{1}{x}, \quad \log xy = \log x + \log y \text{ for positive } x, y \in R;$$

$$\frac{d}{dx} \cos x = -\sin x, \quad \frac{d}{dx} \sin x = \cos x, \quad \cos^2 x + \sin^2 x = 1,$$

$$\sin(x+y) = \sin x \cos y + \cos x \sin y, \quad \cos(x+y) = \cos x \cos y - \sin x \sin y, \text{ for } x, y \in R.$$

By the reduction to classical theorems we also have:

5.5 Theorem¹ (Fubini's Theorem) If $a, b, c, d \in R$, $a \leq b$, $c \leq d$, and $f: [a, b] \times [c, d] \rightarrow R$ is continuous, then

$$\int \int_{[a, b] \times [c, d]} f(x, y) dx dy = \int_a^b dx \int_c^d f(x, y) dy = \int_c^d dy \int_a^b f(x, y) dx.$$

5.6 Theorem¹ (Riemann-Lebesgue Theorem) Let I be a closed interval $[a, b]$ in R . If $f: I \rightarrow R$ is continuous, then for $\beta \in R$

$$\lim_{\alpha \rightarrow \infty} \int_a^b f(t) \sin(\alpha t + \beta) dt = 0.$$

5.7 Theorem¹ (Fourier integral formula) If $f: R \rightarrow R$ is of class C^1 and $\{x \in R \mid f(x) \neq 0\} \subseteq [a, b]$, then for $x \in R$,

$$f(x) = \frac{1}{2\pi} \lim_{\alpha \rightarrow \infty} \int_{-\alpha}^{\alpha} dt \int_{-\infty}^{\infty} f(u) e^{it(u-x)} du.$$

References.

- 1) Bishop, E.: *Foundations of Constructive Analysis*. McGraw Hill 1967.
- 2) Fourman, M. and Scott, D.S.: Sheaves and logic. *Lecture Notes in Mathematics* 753 (1977), Springer-Verlag, pp. 302-401.
- 3) Funayama, N.: On imbedding infinitely distributive lattice completely isomorphically into Boolean algebras, *Nagoya Math. J.* 15 (1959), pp. 71-81.
- 4) Grayson, R.J.: Heyting valued models for intuitionistic set theory. *Lecture Notes in Mathematics* 753 (1977), Springer-Verlag, pp. 402-414.
- 5) Gunning, R. and Rossi, H.: *Analytic Functions of Several Complex Variables*. Prentice-Hall, 1965.
- 6) Hormander, L.: *An Introduction to Complex Analysis in Several Variables*. North-Holland, 1973.
- 7) Rasiowa, H. and Sikorski, R.: *The Mathematics of Metamathematics*. Warsaw, 1963.
- 8) Rousseau, C.: Topos theory and complex analysis. *Lecture Notes in Mathematics* 753 (1977), Springer-Verlag, pp. 623-659.
- 9) Takeuti, G. and Titani, S.: Heyting valued universes of intuitionistic set theory. *Lecture Notes in Mathematics* 891 (1980), Springer-Verlag, pp. 189-306.