

Unordered μ -Baire-like Spaces

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In this paper we define new classes of topological vector spaces which are a generalization of unordered Baire-like spaces, and we obtain a kind of characterization and some results on these spaces.

A locally convex Hausdorff topological vector space is unordered Baire-like, if it cannot be covered by any sequence of rare balanced convex sets [3]. In our previous paper [2], a topological vector space E is called unordered ultra-Baire-like if, for any sequence of subspaces of E which covers E , at least one of the subspaces is both dense and ultrabarrelled. In this paper we shall generalize these concepts furthermore.

Throughout this paper, we assume that each vector topology is of Hausdorff type; E denotes a topological vector space and μ is a real number not less than 2.

A subset A of a vector space is **-convex* if there exists a sequence $\{A_j\}$ of balanced subsets of A such that $A_1 + A_1 \subset A$, $A_{j+1} + A_{j+1} \subset A_j$ and each point of A is absorbed by A_j ($j=1, 2, \dots$) [2]. An ultrabarrel of a topological vector space is a closed, absorbing, balanced and **-convex* set [1] [2]. A topological vector space is ultrabarrelled if and only if every ultrabarrel is a neighborhood of the origin [1].

Definition 1. A subset A of a vector space is called μ -convex if $A + A \subset \mu A$.

It is clear that, if A and B are μ -convex sets, then $A + B$ and kA are μ -convex for any scalar k ; an intersection of μ -convex sets is also μ -convex.

A semiconvex set of a vector space is μ -convex for some μ [1].

Proposition 1. (a) *A convex set is 2-convex.*

(b) *A balanced μ -convex set is λ -convex if $\lambda \geq \mu$.*

(c) *A μ -convex set is semiconvex.*

(d) *A balanced semiconvex set is *-convex.*

Proof. (a), (b) and (c) are trivial by definition. To show (d), let B be a balanced

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semiconvex set. Then $B+B \subset \mu B$ for some μ . Let $B_j = \frac{1}{\mu^j} B$. Then we have $B_{j+1} + B_{j+1} = \frac{1}{\mu^{j+1}} (B+B) \subset \frac{1}{\mu^{j+1}} \mu B = \frac{1}{\mu^j} B = B_j$ and $B_1 + B_1 = \frac{1}{\mu} B + \frac{1}{\mu} B \subset B$. Clearly each B_j is balanced, and each point of B is absorbed by every B_j . Thus B is $*$ -convex.

Definition 2. Let B be a subset of E . B is called a μ -barrel if B is closed, absorbing, balanced and μ -convex. B is called a *semibarrel* if B is closed, absorbing, balanced and semiconvex.

The following proposition is obvious by Proposition 1.

Proposition 2. (a) *A barrel is a μ -barrel.*

(b) *A μ -barrel is a λ -barrel, if $\lambda \geq \mu$.*

(c) *A μ -barrel is a semibarrel.*

(d) *A semibarrel is an ultrabarrel.*

Definition 3. A topological vector space is called μ -barrelled [*semibarrelled*] if each μ -barrel [resp. semibarrel] is a neighborhood of the origin.

By Proposition 2, it is immediate that an ultrabarrelled space is semibarrelled, and that a semibarrelled space is μ -barrelled.

Definition 4. A topological vector space is called *unordered μ -Baire-like* [*unordered semi-Baire-like*] if it cannot be covered by any sequence of rare balanced μ -convex [resp. semiconvex] subsets.

Theorem 1. *A topological vector space E is unordered μ -Baire-like [unordered semi-Baire-like] if and only if, for any sequence of subspaces which covers E , at least one of the subspaces is both dense and μ -barrelled [resp. semibarrelled].*

Proof. We prove this in the unordered μ -Baire-like case only. The other case is proved in the same way.

Suppose that an unordered μ -Baire-like space E is covered by a sequence of subspaces $\{E_n\}$ which is not dense in E or not μ -barrelled. Let B_n be E_n itself if E_n is not dense in E , and let B_n be a μ -barrel in E_n which is not a neighborhood of the origin in E_n if E_n is not μ -barrelled. In either case, B_n is rare and μ -convex in E , and clearly E is covered by the sequence of μ -convex sets $\{mB_n\}$ ($m=1, 2, \dots$; $n=1, 2, \dots$), where each mB_n is rare and μ -convex in E . This is a contradiction.

Conversely, let E be a topological vector space and suppose that, for any sequence of subspaces which covers E , at least one of the subspaces is both dense and μ -barrelled. Let $\{B_n\}$ be a sequence of balanced μ -convex subsets of E which covers E , and E_n be the li-

near span of B_n for each n . Then E is covered by the sequence $\{E_n\}$ and, by the assumption, some E_{n_0} is dense in E and μ -barrelled. Clearly $\overline{B_{n_0}} \cap E_{n_0}$ is a μ -barrel in E_{n_0} and hence it is a neighborhood of the origin in E_{n_0} . As E_{n_0} is dense in E , $\overline{B_{n_0}}$ is a neighborhood of the origin in E . Namely B_{n_0} is not rare in E . Thus E is unordered μ -Baire-like.

By Proposition 2 the following is trivial:

Proposition 3. (a) *An unordered ultra-Baire-like space is unordered semi-Baire-like. Particularly, a Baire topological vector space is unordered semi-Baire-like.*

(b) *An unordered semi-Baire-like space is unordered μ -Baire-like.*

(c) *An unordered μ -Baire-like space is unordered λ -Baire-like, if $\lambda \geq \mu$.*

(d) *An unordered semi-Baire-like space is semibarrelled.*

(e) *An unordered μ -Baire-like space is μ -barrelled.*

We will examine a few permanence properties. In the following, the property “unordered μ -Baire-like” can be replaced always by “unordered semi-Baire-like”.

Proposition 4. *Every quotient space of an unordered μ -Baire-like space is unordered μ -Baire-like.*

Proof. Let E be an unordered μ -Baire-like space, F be a closed subspace of E and q be the quotient mapping of E onto E/F . Suppose that E/F is covered by a sequence $\{A_n\}$ of closed balanced μ -convex subsets. Then $q^{-1}(A_n)$ is a closed balanced μ -convex subset of E and E is covered by the sequence $\{q^{-1}(A_n)\}$. Then by the assumption some $q^{-1}(A_{n_0})$ is not rare in E . Thus $A_{n_0} = q(q^{-1}(A_{n_0}))$ is not rare in E/F . Therefore E/F is unordered μ -Baire-like.

Proposition 5. *Let E be a topological vector space and F be a dense subspace of E . If F is unordered μ -Baire-like, then E is also unordered μ -Baire-like.*

Proof. Let $\{A_n\}$ be a sequence of closed balanced μ -convex subsets of E which covers E . Then $\{A_n \cap F\}$ is a sequence of closed balanced μ -convex subsets of F which covers F . By the assumption, for some n_0 , $A_{n_0} \cap F$ is not rare in F . Then there are a neighborhood U of the origin in F and an element a in $A_{n_0} \cap F$ such that $a + U \subset A_{n_0} \cap F$. As A_{n_0} is balanced and μ -convex, we have

$$U \subset (a + U) - (a + U) \subset A_{n_0} + A_{n_0} \subset \mu A_{n_0}.$$

Thus $\frac{1}{\mu}U \subset A_{n_0}$ and $A_{n_0} \cap F$ is a neighborhood of the origin in F . Then there exists an open neighborhood O of the origin in E such that $O \cap F$ is contained in $A_{n_0} \cap F$. For an arbitrary element u of O , there is a neighborhood V of the origin in E such that $u + V \subset O$ as O is open. Since F is dense in E , for every neighborhood W of the origin in E , $u + V \cap W$ intersects with F . Hence, $u + W$ and $O \cap F$ have a non-empty intersection. Thus O is

contained in the closure of $O \cap F$, and O is contained in A_{n_0} . Therefore A_{n_0} is a neighborhood of the origin in E and this proves that E is unordered μ -Baire-like.

Corollary. *The completion of an unordered μ -Baire-like space is unordered μ -Baire-like.*

Proposition 6. *Every countable codimensional subspace of an unordered μ -Baire-like space unordered μ -Baire-like.*

This proposition is proved in a similar way as the proof of Theorem 4.4 of [3].

Finally we have the following results on a locally convex case in the same way as Theorem 2 in [2] by Proposition 2.

Proposition 7. *Every unordered μ -Baire-like [semi-Baire-like] space is unordered Baire-like for the finest locally convex topology which is coarser than the original topology.*

Corollary. *Every locally convex unordered μ -Baire-like [semi-Baire-like] space is unordered Baire-like.*

References

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| 1) S. O. Iyahan, <i>On certain classes of linear topological spaces</i> , Proc. London Math. Soc. (3) 18 (1968), 285-307 | 19 (1983), 89-92 |
| 2) N. Kitsunezaki, K. Kera and Y. Terao, <i>On unordered ultra-Baire-like spaces</i> , TRU Math. | 3) A. R. Todd and S. A. Saxon, <i>A property of locally convex Baire spaces</i> , Math. Ann. 206 (1973), 23-34. |