

Strength of Columns under Biaxially Eccentric Loads

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This paper deals with the simple design method for reinforced concrete short and slender columns subjected to biaxially eccentric loads.

Firstly, two calculating methods for the ultimate capacity of the concrete columns are proposed. One is the method proposed for computer analysis and the other for practical simple design.

Next, the comparisons of two proposed methods with the results of tests on actual thirty columns provide the confirmation of the validity of these methods.

These methods are recommended for general use.

I Introduction

In the ultimate strength design of the columns, it is quite important to examine not only the ultimate strength but the ultimate deflection at the same time, especially, for the columns under biaxially eccentric loads.

This paper reports the analysis and the results of tests on the strength and the deflection of reinforced concrete short and slender columns subjected to biaxially eccentric loads.

Firstly, a calculating method for the ultimate capacity of the concrete columns is proposed. This method consists of two analytical programs easily available for computer: one is the analysis of failure envelope of column cross-section and the other is the analysis of column deflection under biaxially eccentric load. The strength and the deflection at ultimate stage can be predicted by combining these two methods.

Based on the analysis and the results of tests, a simple design procedure for slender columns subjected to biaxially eccentric loads is proposed.

II Theoretical analysis

(1) Calculation of failure envelope

General assumptions made in the analysis are:

- i) Plane sections remain plane before and after bending.
- ii) No slip occurs between steel and concrete.
- iii) Concrete does not resist any tension.
- iv) Stress distributions in concrete at ultimate stage are shown in Fig. 1.
- v) Maximum strain in concrete is limited, for example, to 0.003.

For a given cross-section, the ultimate eccentricity and the ultimate axial load can be determined if the location of the neutral axis is fixed. Therefore if the neutral axis

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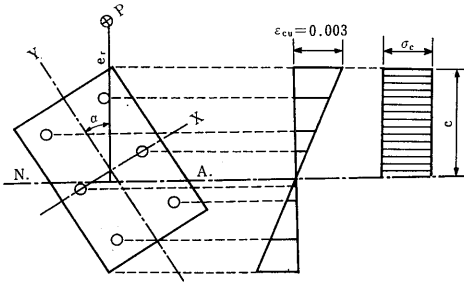


Fig. 1 Stress distribution in concrete at ultimate stage

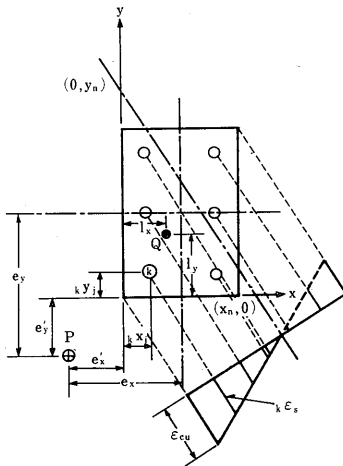


Fig. 2 Cross-section of column

where σ_i = stress in element i

A_i = area of element i

y_i = distance from x-axis to element i

e'_y = distance from x-axis to axial load P

The strain in element i can be determined from the following:

$$\epsilon_i = \left(1 - \frac{y_i}{H_y}\right) \cdot \epsilon_{y_2} + \frac{y_i}{H_y} \cdot \epsilon_{y_1} \quad (6)$$

where H_y , ϵ_{y_1} and ϵ_{y_2} are shown in Fig. 5. And by using the stress-strain relationship,

$$\sigma_i = E_i \cdot \epsilon_i \quad (7)$$

where E_i is the secant modulus of elasticity of element i .

Substituting eqs. (6) and (7) in eqs. (4) and (5), we obtain the following equilibrium equations:

$$\sum_{i=1}^n \left(1 - \frac{y_i}{H_y}\right) \cdot E_i \cdot A_i \cdot \epsilon_{y_2} + \sum_{i=1}^n \frac{y_i}{H_y} \cdot E_i \cdot A_i \cdot \epsilon_{y_1} + P = 0 \quad (8)$$

is arbitrarily assumed, as shown in Fig. 2, the following equilibrium conditions need be satisfied.

$$P = C + \sum_{k=1}^n k E_s \cdot k A_s \cdot k \epsilon_s(i, j) \quad (1)$$

$$P \cdot e'_x = C \cdot l_x + \sum_{k=1}^n k E_s \cdot k A_s \cdot k \epsilon_s(i, j) \cdot x_i \quad (2)$$

$$P \cdot e'_y = C \cdot l_y + \sum_{k=1}^n k E_s \cdot k A_s \cdot k \epsilon_s(i, j) \cdot y_j \quad (3)$$

where P = ultimate axial load

$k E_s$ = Young's modulus of steel k

$k A_s$ = area of steel k

C = compressive force on concrete

$k \epsilon_s(i, j)$ = strain in steel k

The calculated ultimate strength surfaces of square and rectangular columns are shown, as examples, in Figs. 3 and 4, respectively.

(2) Analysis of slender reinforced concrete column

The divided elements of section and the shape of stress-strain relationships are shown in Figs. 5 and 6, respectively.

For y direction, the following equilibrium conditions need be satisfied.

$$\sum_{i=1}^n \sigma_i \cdot A_i + P = 0 \quad (4)$$

$$\sum_{i=1}^n \sigma_i \cdot A_i \cdot y_i + P \cdot e'_y = 0 \quad (5)$$

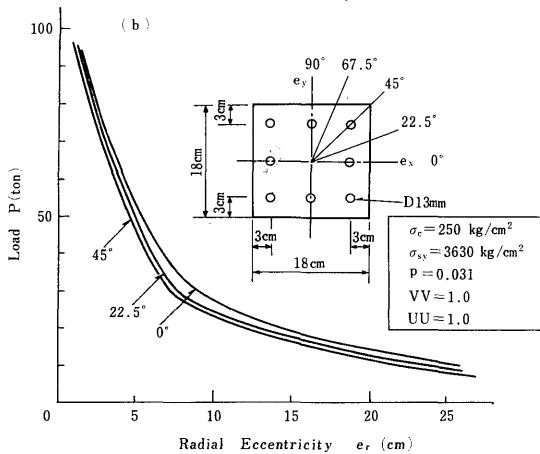
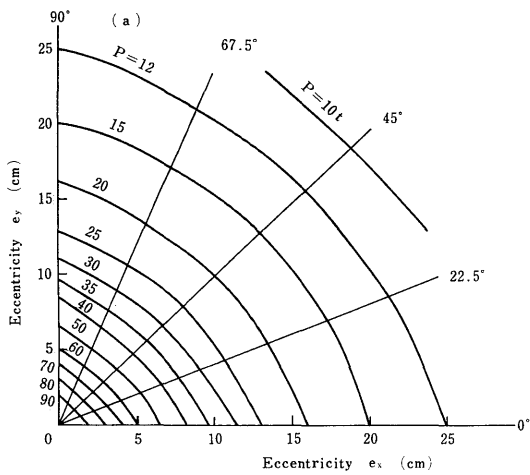


Fig. 3 Failure surface of square column

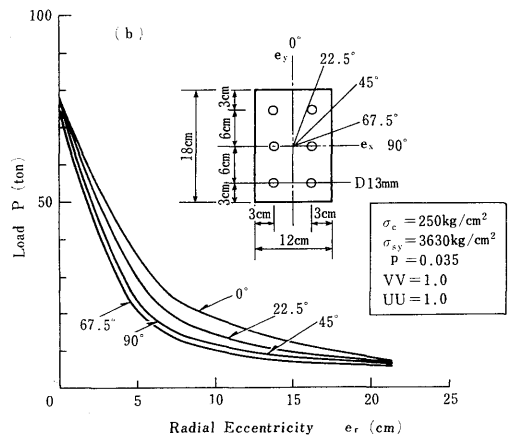
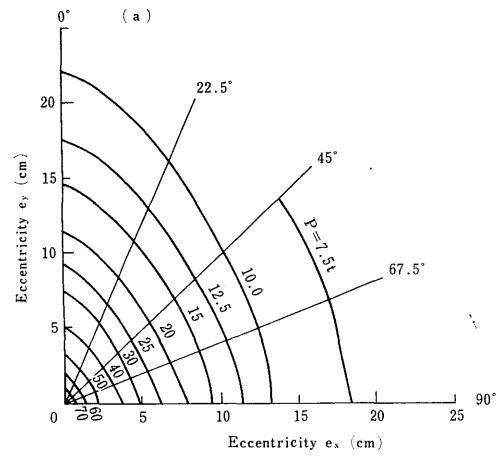


Fig. 4 Failure surface of rectangular column

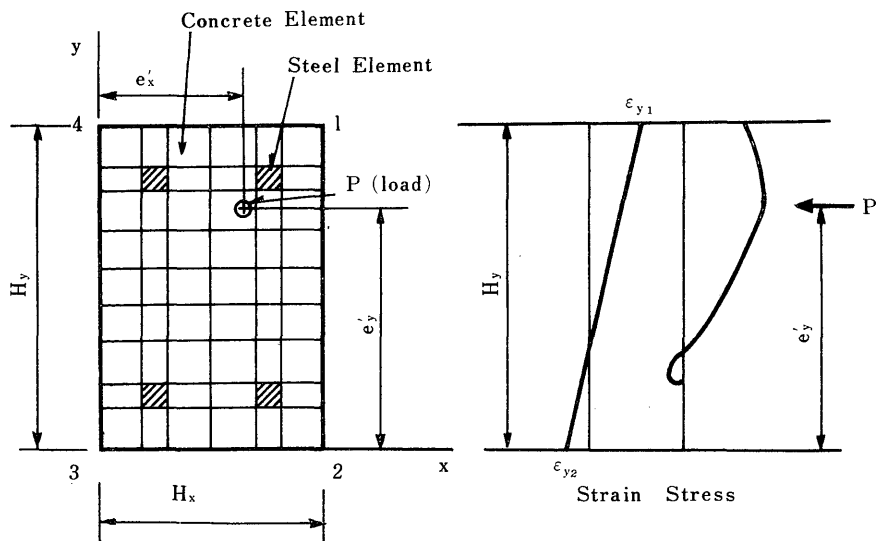


Fig. 5 Divided elements of the cross-section

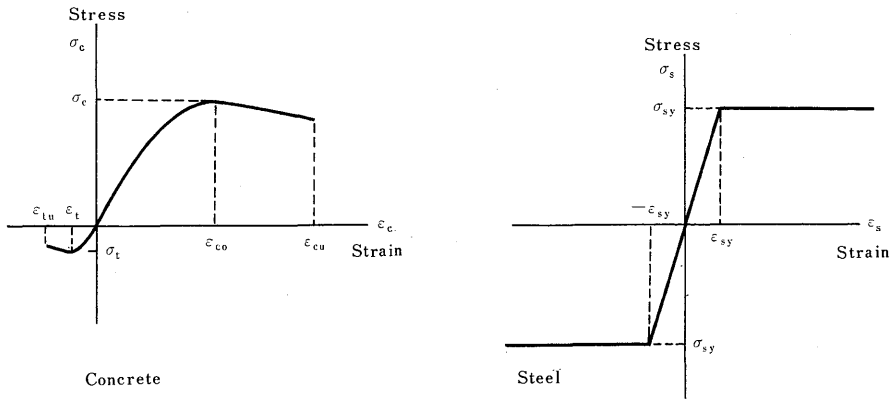


Fig. 6 Stress-strain relationships for concrete and steel

$$\sum_{i=1}^n \left(1 - \frac{y_i}{H_y} \right) \cdot y_i \cdot E_i \cdot A_i \cdot \varepsilon_{y2} + \sum_{i=1}^n \frac{y_i^2}{H_y} \cdot E_i \cdot A_i \cdot \varepsilon_{y1} + P \cdot e'_y = 0 \quad (9)$$

Then, from eqs. (8) and (9), the following expressions for the fiber strains are obtained.

$$\varepsilon_{y1} = P \cdot \frac{c - a \cdot e'_y}{ad - bc} \quad (10)$$

$$\varepsilon_{y2} = P \cdot \frac{b \cdot e'_y - d}{ad - bc} \quad (11)$$

where

$$a = \sum_{i=1}^n \left(1 - \frac{y_i}{H_y} \right) \cdot E_i \cdot A_i, \quad b = \sum_{i=1}^n \frac{y_i}{H_y} \cdot E_i \cdot A_i$$

$$c = \sum_{k=1}^n \left(1 - \frac{y_i}{H_y} \right) \cdot y_i \cdot E_i \cdot A_i, \quad d = \sum_{i=1}^n \frac{y_i^2}{H_y} \cdot E_i \cdot A_i$$

For x direction the similar relations can be derived for ε_{x1} and ε_{x2} .

Next, the slender column is divided into several numbers of small segments in the longitudinal direction as shown in Fig. 7. And the deflection under the elastic load is determined for each segment. Adding the deflection to the original eccentricity gives the new eccentricity value.

Thus, the repetition and integration methods are adopted for determining the final deflection of a column. The basic program is outlined in Fig. 8.

According to this method, we can easily obtain the change of the eccentricities from the initial to the final locations at each load level and the strength for radial direction of the column under biaxially eccentric loads.

(3) Prediction of ultimate strength and deflection of column under biaxially eccentric loads

Considering two calculation methods as described above, the ultimate capacity of reinforced concrete slender columns under biaxially eccentric loads can be determined as the intersection of the deflection curve with the ultimate strength envelope as shown in Fig. 9.

The analytical method ever proposed by many investigators^{1)~9)} are often based

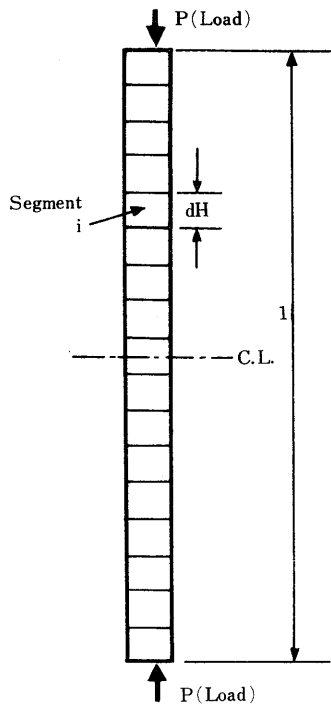


Fig. 7 Divided segments

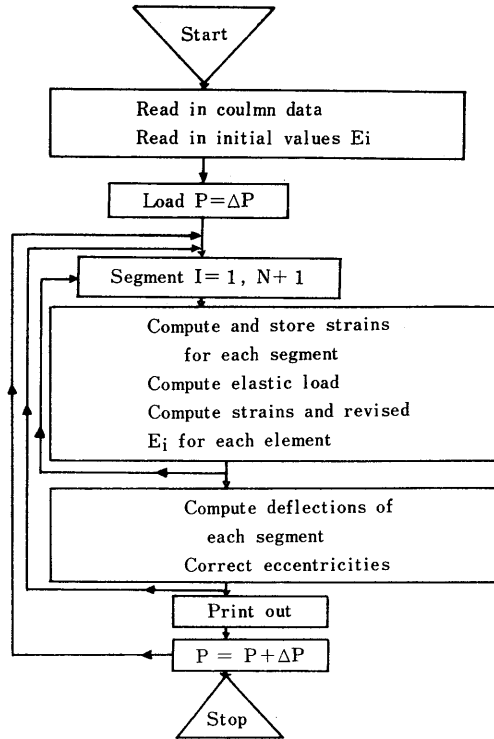


Fig. 8 Basic program of analysis

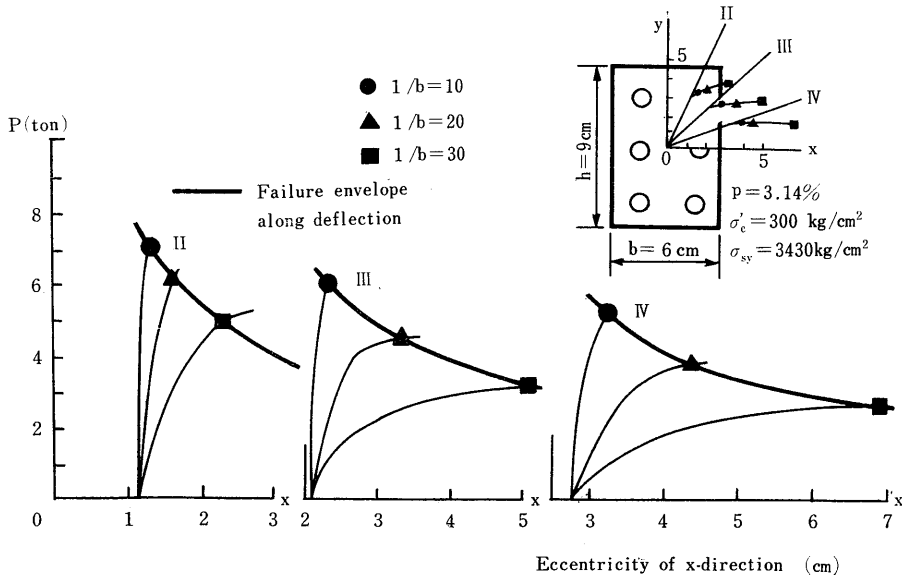


Fig. 9 Load-eccentricity relationship at the center section of column and failure envelope derived from failure surface

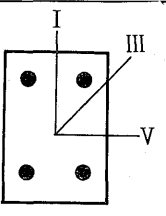
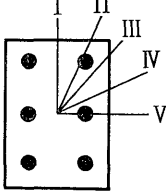
on the assumed direction or fixed eccentricity as to the ultimate deflection, but this method has characteristics of dealing with the change of the location of the eccentricity due to biaxially eccentric loads.

III Test of the columns under biaxially eccentric loads

(1) Test programs

Totally sixty-two short and slender columns were made and tested,⁷⁾ and the test programs for slender columns are shown in Table 1.

Table 1 Test programs

		Column	Cross-section	Column length(cm)	Eccentricity direction	
Program I	A Series	A 60 I A 60 III A 60 V	A	60	I III V	 III = 45°
		A 120 I A 120 III A 120 V	A	120	I III V	
		A 180 I A 180 III A 180 V	A	180	I III V	
	B Series	B 60 I B 60 II B 60 III B 60 IV B 60 V	B	60	I II III IV V	 II = 22.5° III = 45° IV = 67.5°
		B 120 I B 120 II B 120 III B 120 IV B 120 V	B	120	I II III IV V	
		B 180 I B 180 II B 180 III B 180 IV B 180 V	B	180	I II III IV V	
	Program II	A 60 III' A 120 III' A 180 III'	A	60 120 180	III	
		B 60 III' B 120 III' B 180 III'	B	60 120 180	III	

All of slender columns have the same rectangular cross-section of 6×9 cm and the same initial eccentricity of 3 cm. Columns of series A and B were reinforced with four and six 6 mm bars, respectively. The column length of each group in two series were 60, 120, and 180 cm, respectively. Initial directions of eccentricities in a group were I, III and V or I, II, III, IV and V as shown in Table 1.

(2) Results of tests and discussion for deflection

Fig. 10 shows the comparison of the measured deflections with the computed ones. The stress-strain relationship of concrete is assumed in the calculated also as shown in Fig. 10.

Fig. 11 clearly shows the change of eccentricities with increasing load from the

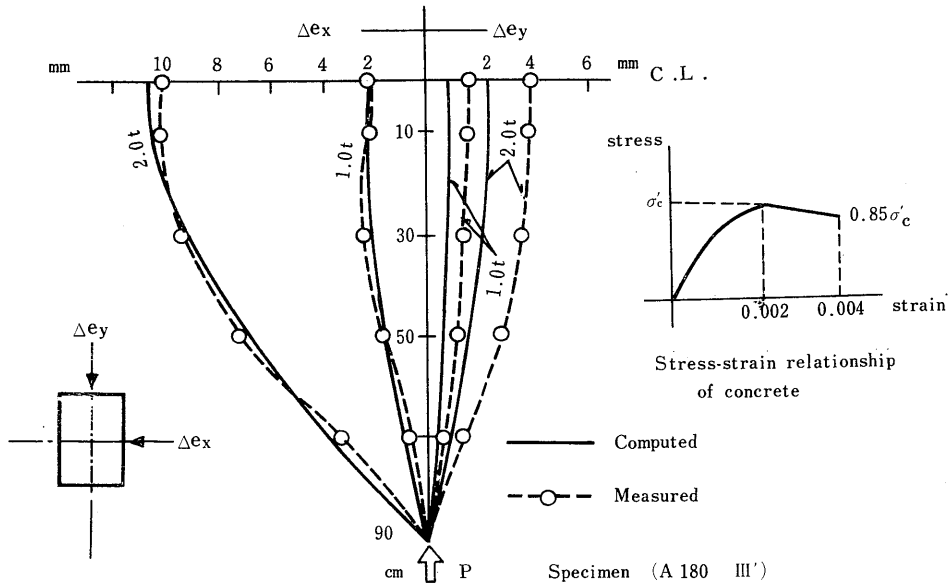


Fig. 10 Comparison of the measured deflections with the computed ones

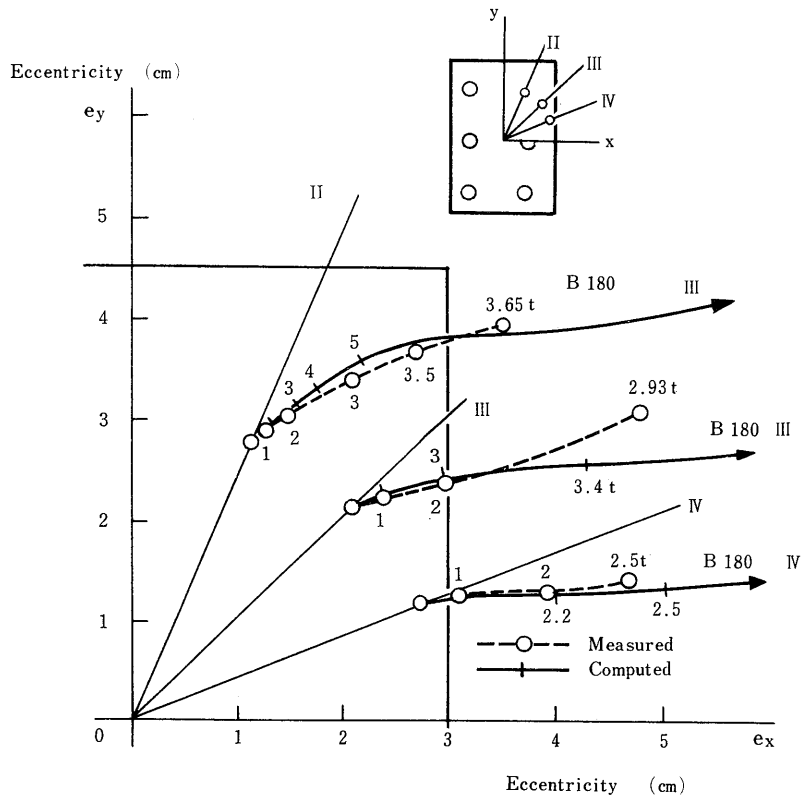


Fig. 11 Load-eccentricity relationships in the mid-height section of column

initial to the final locations, the final eccentricities being not located along the inclination of initial direction.

The torsional deformations due to the biaxially eccentric load are not considered

here, but it should be studied as an important problem in future.

IV Simple design solutions for the ultimate deflection and strength under biaxially eccentric loads

(1) Ultimate deflection and strength

A slender column when subjected to biaxially eccentric load deflects as already shown in Figs. 10 or 11.

Fig. 12 is also one measurement showing how the initial eccentricity of load on the critical section actually varies with increasing load for each of five slender columns subjected to uniaxially or biaxially eccentric load. The initial (e_{r0}) and final eccentricities ($e_{r0} + \Delta e_r$) projected on the $e_x - e_y$ plane is shown in Fig. 13(a). The final eccentricities are located on the inclination (β) deviated from the initial direction (α), and $P - \beta$ relationships are replotted in Fig. 13(b). Radial ultimate moments based on the final eccentricity are calculated from Fig. 13(a) and Fig. 13(b), and shown in Fig. 13(c).

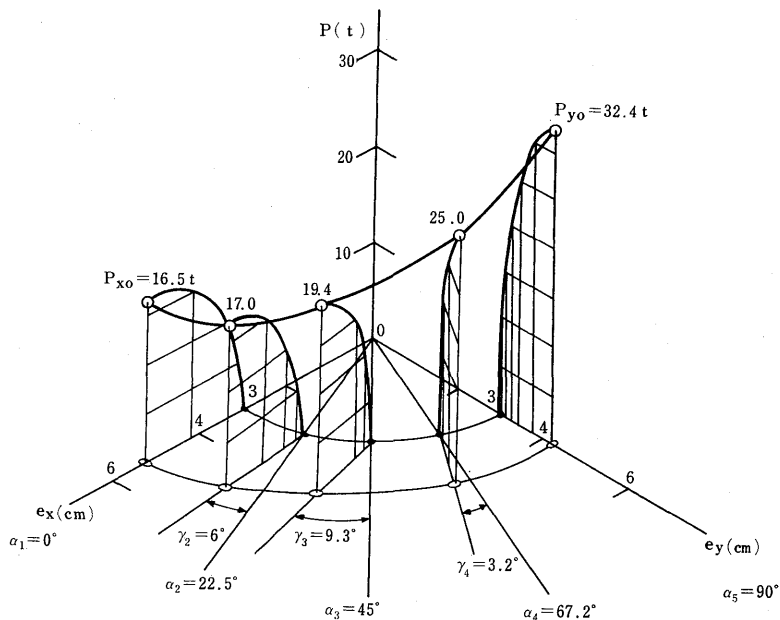


Fig. 12 Load-eccentricity relationships at the center section of biaxially loaded columns

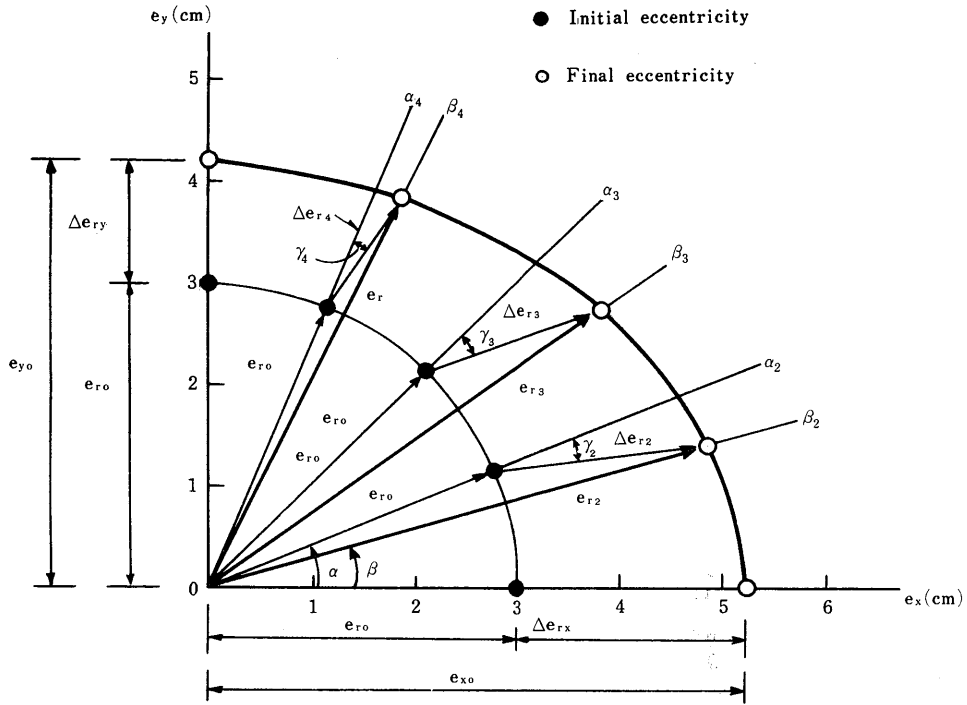
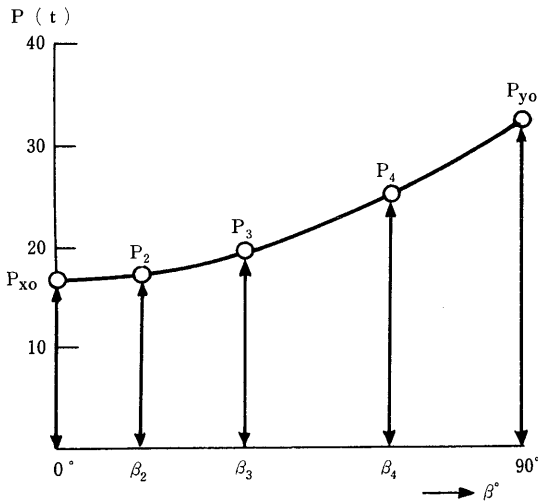
(2) Modified additional moment method

Detailed calculations as given in II, III are rather complicated, so the simple design procedure is proposed.

It may be deduced that the radial eccentricity $Q(e_x, e_y)$ at the final stage can be approximated by the following equation, as shown in Fig. 14:

$$\left(\frac{e_x}{e_{x0}}\right)^2 + \left(\frac{e_y}{e_{y0}}\right)^2 = 1 \quad (12)$$

The actual eccentricity e_x , or e_y is given by

Fig. 13(a) Initial and final eccentricities projected on the e_x — e_y planeFig. 13(b) $P - \beta$ relationship

$$e_x = e_{x0} \cdot \cos \alpha$$

$$e_y = e_{y0} \cdot \sin \alpha$$

where:

α and β are the inclinations of the lines joining the initial and final loading points (eccentricities) to the centroid of the column cross-section, respectively, and e_{x0} and e_{y0} are the final values of uniaxial eccentricity along x and y axis, respectively.

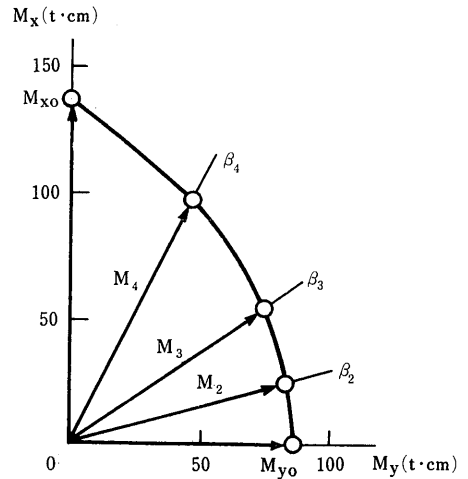


Fig. 13(c) Radial ultimate moments on final direction of eccentricity

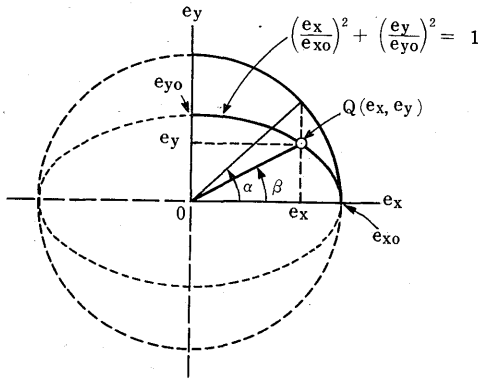


Fig. 14 Supposed radial eccentricity

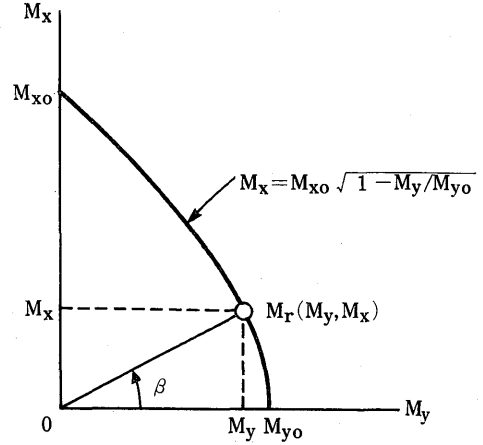


Fig. 15 Approximation for radial ultimate moment

Actual radial eccentricity at the ultimate stage is given by eq. (13):

$$e_r = \sqrt{e_x^2 + e_y^2} \quad (13)$$

and the relation between α and β is given as follows:

$$\beta = \tan^{-1} \left(\frac{e_{y0}}{e_{x0}} \tan \alpha \right) \quad (14)$$

Next, it may also be assumed that the outer limits of the ultimate moment corresponding to Fig. 14 is approximately given by the expression as shown in Fig. 15.

$$M_x = M_{x0} \sqrt{1 - \frac{M_y}{M_{y0}}} \quad (15)$$

where M_{y0} and M_{x0} are the equivalent uniaxial moments for the ultimate moment about x - and y -axis, respectively, and M_y and M_x are the components of the actual ultimate radial moment M_r .

Equation (15) gives the following relations,

$$M_y = M_r \cos \beta \quad (16)$$

$$M_x = M_r \sin \beta \quad (17)$$

$$M_r = \frac{M_{x0}}{A \sin \beta} [\sqrt{A+1} - 1] \quad (18)$$

$$\text{where } A = 2 \cdot \frac{M_{y0}}{M_{x0}} \cdot \tan \beta. \quad (19)$$

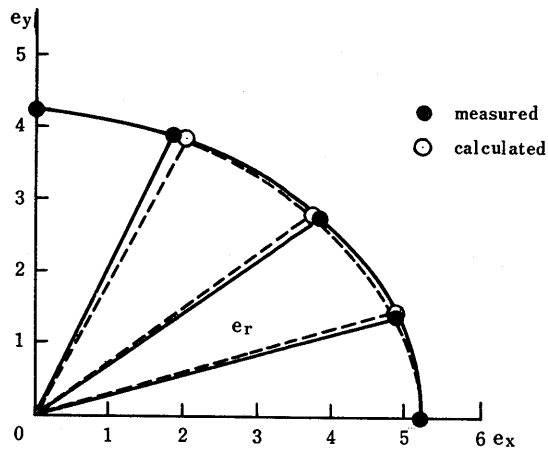
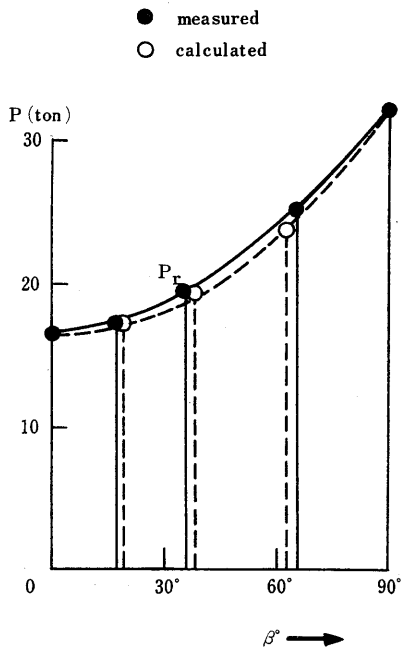
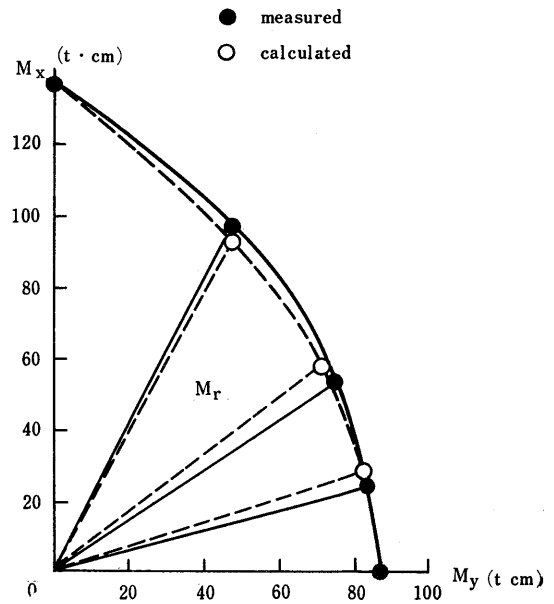
Thus, from equations (13) and (18), the ultimate load P under biaxially eccentric load can be given by

$$P_r = \frac{M_r}{e_r} \quad (20)$$

(3) Comparison with tests

The detailed comparison of the proposed simple design procedures with the results of tests on actual five columns are shown, for example, in Figs. 16 (a), (b) and (c), concerning the ultimate deflection (e_r), the ultimate load (P_r) and the ultimate moment (M_r), respectively. In these figures, for uniaxial ultimate loads, deflection and moments, the measured values are used instead of the calculated values.

If the uniaxial ultimate deflection and load are calculated, for general use, by the

Fig. 16(a) Comparison for e_r Fig. 16(b) Comparison for P_r Fig. 16(c) Comparison for M_r

additional moment method introduced by Prof. Macgregor in the Preliminary Report Theme II⁸⁾ (B.S. or FIP-CEB recommendation), the comparison of the author's method with the results of tests is shown in Table 2.

Then, the simple solution proposed here for the ultimate capacities of slender columns subjected to biaxially eccentric load may be used for design purpose.

Table 2 Comparison between test results with calculated results

Column	l cm	b cm	h cm	d/h	σ'_c kg/cm ²	σ_{sy} kg/cm ²	e_{ro} cm	e_r test cm	e_r cal cm	$\frac{e_r \text{ test}}{e_r \text{ cal}}$	P_r test ton	P_r cal ton	$\frac{P_r \text{ test}}{P_r \text{ cal}}$	Remark
A 60 I III V	60	6	9	0.833	296	3430	3	3.30 3.30 3.47	3.16 3.22 3.29	1.044 1.025 1.055	7.75 5.35 3.95	8.59 4.36 3.54	0.902 1.227 1.116	
A 120 I III V	120	6	9	0.833	296	3430	3	4.20 4.50 4.17	3.65 3.88 4.10	1.151 1.160 1.017	7.40 3.65 2.68	7.96 3.10 2.37	0.930 1.178 1.131	
A 180 I III V	180	6	9	0.833	296	3430	3	5.46 5.12 5.42	4.57 4.99 5.37	1.195 1.026 1.010	6.00 2.45 2.09	7.02 2.11 1.56	0.855 1.161 1.340	
B 60 I II III IV V	60	6	9	0.833	318	3430	3	3.35 3.23 3.27 3.50 3.59	3.16 3.18 3.22 3.27 3.28	1.060 1.016 1.016 1.070 1.095	8.87 5.40 4.66 4.40 4.69	9.06 6.97 5.97 5.51 5.39	0.979 0.775 0.781 0.799 0.870	
B 120 I II III IV V	120	6	9	0.833	293	3430	3	4.34 4.48 4.50 4.35 4.01	3.66 3.73 3.89 4.04 4.10	1.186 1.201 1.157 1.077 0.978	6.20 4.33 4.16 3.35 3.00	8.02 5.52 4.30 3.72 3.55	0.773 0.784 0.967 0.901 0.845	
B 180 I II III IV V	180	6	9	0.833	318	3430	3	5.11 5.22 5.45 6.45 6.25	4.58 4.70 4.97 5.24 5.34	1.116 1.111 1.097 1.231 1.170	5.05 3.65 2.93 2.90 2.73	7.28 4.33 3.06 2.52 2.37	0.694 0.843 0.958 1.151 1.152	
A 60 A 120 A 180	60 120 180	6	9	0.833	333	3430	3	3.25 4.16 5.55	3.22 3.87 4.96	1.010 1.075 1.119	4.28 4.10 2.44	4.50 3.18 2.16	0.951 1.289 1.130	
B 160 B 120 B 180	160 120 180	6	9	0.833	333	3430	3	3.38 4.25 5.55	3.22 3.88 4.96	1.050 1.095 1.119	4.75 4.40 3.00	6.05 4.44 3.09	0.785 0.991 0.971	
Average										1.091			0.974	

V Conclusions

Two calculating methods of the ultimate capacity of concrete columns with taking into consideration the column deflection under biaxially eccentric loads are proposed.

One is primarily for computer analysis, and it can analyze the complicated behavior of the biaxially eccentric loaded columns.

The other method is proposed for practical design, and it is developed and refined to achieve both simplicity in use and, as far as possible, a realistic representation of actual behavior.

The theoretical ultimate loads and deflections calculated by both methods agree well with the results of sixty-two model columns.

These methods are recommended for general use.

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