

Numerical Analysis of Three Dimensional Boundary Layer I

— Separation of the Layer with Constant Adverse Pressure Gradient

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The formulas for an approximate solution are deduced by the moment-method. These formulas are available to solve numerically the boundary layer equations of three dimensional incompressible steady flow over a body surface. Then the first case of application is given, which is one of the simplest where the boundary layer is resisted by a constant pressure gradient. Three flow patterns near the separation region are obtained. One of them is the so-called singular separation flow, and the others ordinary ones. These ordinary separation flows are distinguished, one from the other according to the behaviour of the secondary flow. That the flow patterns of some ordinary separation have a good resemblance to the singular separation flow, is shown.

1. Introduction

In a three dimensional boundary layer, in general, the velocity must be described by two components. Thus the two dimensional definition of separation, the vanishing of one of skin friction components, may sometimes mean a local change of the flow direction at the surface. Therefore it is necessary to establish the criterion for the three dimensional boundary layer separation.

Many qualitative concepts have been proposed, but few quantitative ones. The leading ideas concerning the separation of three dimensional flow, first proposed by Prandtl,¹⁾ are based on the examination of the pattern of the limiting streamlines on the body surface. By the branching loci of limiting streamlines, Maskell²⁾ distinguished the ordinary separations from the singular one. In the topological study on separation, Lighthill³⁾ proposed two singularities, nodal point and saddle point, and cylindrical point as the third situation among the net work of limiting streamlines. He identified further a separation line as the line which joins a saddle point to a nodal point of separation. Cylindrical points always occur as a line of such points.

Maskell's criterion of limiting streamlines or Lighthill's of the line joining two singularities has improved our understanding of the separation, but now it may be still adequate. More informations, however, of the flow patterns near the separation region in the internal flow have to be brought to establish the complete determination of the three dimensional boundary layer separation. In particular, experimental measurements

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or numerical results are very useful.⁴⁾

In this paper, solving the three dimensional boundary layer equations over a flat plate, from the leading edge to the separation region, though the accuracy of approximation of the boundary layer equations may decrease in such a region, the relationship between parameters concerning the outer flow and the flow patterns in the separation region has been studied. Three representative flow patterns of the separation region have been found, which may be classified as follows: 1) singular separation points occur as a continuous line of such points, 2) the direction of the cross flow near the plate and that at the outer edge are the same, 3) these directions are opposite.

The calculation method used in this paper is called the moment-method, which was proposed and developed by Yamada⁵⁾ and others.

2. Basic equations and transformations

The boundary layer equations of three dimensional steady incompressible flow over a flat plate are

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} + \nu \frac{\partial^2 u}{\partial z^2}, \quad (1-a)$$

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = U \frac{\partial V}{\partial x} + V \frac{\partial V}{\partial y} + \nu \frac{\partial^2 v}{\partial z^2}, \quad (1-b)$$

and the continuity equation is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \quad (2)$$

where u , v , and w are the velocity components in x , y , and z directions of Cartesian coordinates respectively. The plate forms the half-plane, $z=0$, $x \geq 0$. U and V are velocity components of the external inviscid flow in x and y directions.

The boundary conditions for u , v , and w are

$$\begin{aligned} z=0 : u=v=w=0, \\ z \rightarrow \infty : u \rightarrow U, v \rightarrow V. \end{aligned} \quad (3)$$

Let $\delta(x, y)$ be the boundary layer thickness, a a specific length, and U a specific velocity. Introducing the new dimensionless coordinates

$$\xi = \frac{x}{a}, \quad \eta = \frac{y}{a}, \quad \zeta = \frac{z}{\delta}, \quad (4)$$

and non-dimensional quantities

$$\begin{aligned} u' = \frac{u}{U_0}, \quad v' = \frac{v}{U_0}, \quad w' = \frac{w}{U_0}, \\ U' = \frac{U}{U_0}, \quad V' = \frac{V}{U_0}, \quad \Delta = \frac{\delta}{a}, \end{aligned} \quad (5)$$

we have the following transforms:

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{1}{a} \left(\frac{\partial}{\partial \xi} - \frac{\zeta}{2X} \frac{\partial X}{\partial \xi} \frac{\partial}{\partial \zeta} \right), \quad \frac{\partial}{\partial y} = \frac{1}{a} \left(\frac{\partial}{\partial \eta} - \frac{\zeta}{2X} \frac{\partial X}{\partial \eta} \frac{\partial}{\partial \zeta} \right), \\ \frac{\partial}{\partial z} &= \frac{1}{a\Delta} \frac{\partial}{\partial \zeta}, \quad \frac{\partial^2}{\partial z^2} = \left(\frac{1}{a\Delta} \right)^2 \frac{\partial^2}{\partial \zeta^2}, \end{aligned} \quad (6)$$

where $X = R\Delta^2$, R is a Reynolds number:

$$R = \frac{U_0 a}{\nu} \quad (7)$$

and ν is the kinematic viscosity.

Eliminating w from Eqs. (1) and (2), and using the transformations introduced above, we have

$$\begin{aligned} L_1(u', v', X) = & u' \left(\frac{\partial u'}{\partial \xi} - \frac{\zeta}{2X} \frac{\partial X}{\partial \xi} \frac{\partial u'}{\partial \zeta} \right) + v' \left(\frac{\partial u'}{\partial \eta} - \frac{\zeta}{2X} \frac{\partial X}{\partial \eta} \frac{\partial u'}{\partial \zeta} \right) \\ & + \frac{\partial u'}{\partial \zeta} \int_0^\zeta \left(-\frac{\partial u'}{\partial \xi} + \frac{\zeta}{2X} \frac{\partial X}{\partial \xi} \frac{\partial u'}{\partial \zeta} - \frac{\partial v'}{\partial \eta} + \frac{\zeta}{2X} \frac{\partial X}{\partial \eta} \frac{\partial v'}{\partial \zeta} \right) d\zeta \\ & - \left(U' \frac{\partial U'}{\partial \xi} + V' \frac{\partial U'}{\partial \eta} \right) - \frac{1}{X} \frac{\partial^2 u'}{\partial \zeta^2} = 0, \end{aligned} \quad (8-a)$$

$$\begin{aligned} L_2(u', v', X) = & u' \left(\frac{\partial v'}{\partial \xi} - \frac{\zeta}{2X} \frac{\partial X}{\partial \xi} \frac{\partial v'}{\partial \zeta} \right) + v' \left(\frac{\partial v'}{\partial \eta} - \frac{\zeta}{2X} \frac{\partial X}{\partial \eta} \frac{\partial v'}{\partial \zeta} \right) \\ & + \frac{\partial v'}{\partial \zeta} \int_0^\zeta \left(-\frac{\partial u'}{\partial \xi} + \frac{\zeta}{2X} \frac{\partial X}{\partial \xi} \frac{\partial u'}{\partial \zeta} - \frac{\partial v'}{\partial \eta} + \frac{\zeta}{2X} \frac{\partial X}{\partial \eta} \frac{\partial v'}{\partial \zeta} \right) d\zeta \\ & - \left(U' \frac{\partial V'}{\partial \xi} + V' \frac{\partial V'}{\partial \eta} \right) - \frac{1}{X} \frac{\partial^2 v'}{\partial \zeta^2} = 0, \end{aligned} \quad (8-b)$$

where the boundary conditions (3) become now in the form

$$\begin{aligned} \zeta = 0 : u' = v' = 0, \\ \zeta = 1 : u' = U', \quad v' = V'. \end{aligned} \quad (9)$$

3. Approximate equations

In this paper, the moment-method due to Yamada and others, which is one of the weighted residuals method, is employed. An approximate solution of the boundary layer Eq. (8) is assumed in the form

$$\bar{u}(\xi, \eta, \zeta) = \sum_{i=0}^5 u_i(\xi, \eta) \zeta^i, \quad \bar{v}(\xi, \eta, \zeta) = \sum_{j=0}^5 v_j(\xi, \eta) \zeta^j, \quad (10)$$

where the set $(1, \zeta, \zeta^2, \zeta^3, \dots)$ constitutes a complete set in $0 \leq \zeta \leq 1$, which corresponds to the boundary layer thickness. The expansion coefficients u_i and v_j are to be determined, some of which by the boundary conditions (9) and newly added ones: in all

$$\begin{aligned} \zeta = 0 : \bar{u} = 0, \quad \frac{\partial^2 \bar{u}}{\partial \zeta^2} = -X \left(U' \frac{\partial U'}{\partial \xi} + V' \frac{\partial U'}{\partial \eta} \right), \\ \bar{v} = 0, \quad \frac{\partial^2 \bar{v}}{\partial \zeta^2} = -X \left(U' \frac{\partial V'}{\partial \xi} + V' \frac{\partial V'}{\partial \eta} \right), \\ \zeta = 1 : \bar{u} = U', \quad \frac{\partial \bar{u}}{\partial \zeta} = 0, \quad \frac{\partial^2 \bar{u}}{\partial \zeta^2} = 0, \\ \bar{v} = V', \quad \frac{\partial \bar{v}}{\partial \zeta} = 0, \quad \frac{\partial^2 \bar{v}}{\partial \zeta^2} = 0, \end{aligned} \quad (11)$$

and the others by the requirement of vanishing first three moments:

$$\begin{aligned} \int_0^1 L_1(\bar{u}, \bar{v}, X) \zeta^n d\zeta = 0, \quad n=0, 1, \\ \int_0^1 L_2(\bar{u}, \bar{v}, X) \zeta^m d\zeta = 0, \quad m=0. \end{aligned} \quad (12)$$

Our approximate solutions (10), after satisfaction of the boundary conditions (11), may be written as

$$\begin{aligned} \bar{u}(\xi, \eta, \zeta) = U' F(\zeta) + X A \cdot G(\zeta) + \vartheta H(\zeta), \\ \bar{v}(\xi, \eta, \zeta) = V' F(\zeta) + X B \cdot G(\zeta) + \phi H(\zeta), \end{aligned} \quad (13-a)$$

where

$$\begin{aligned} F(\zeta) &= 2\zeta - 2\zeta^3 + \zeta^4, \quad G(\zeta) = \zeta - 3\zeta^2 + 3\zeta^3 - \zeta^4, \quad H(\zeta) = -\zeta + 6\zeta^3 - 8\zeta^4 + 3\zeta^5; \\ A(\xi, \eta) &= \frac{1}{6} \left(U' \frac{\partial U'}{\partial \xi} + V' \frac{\partial U'}{\partial \eta} \right), \quad B(\xi, \eta) = \frac{1}{6} \left(U' \frac{\partial V'}{\partial \xi} + V' \frac{\partial V'}{\partial \eta} \right); \\ \vartheta(\xi, \eta) &= u_s(\xi, \eta)/3, \quad \varphi(\xi, \eta) = v_s(\xi, \eta)/3. \end{aligned} \quad (13-b)$$

Two parameters θ , φ , and the boundary layer thickness parameter X , which remain undetermined, have to be determined now by the moment equations.

Thus we obtain

$$\begin{aligned} a_n X \frac{\partial \vartheta}{\partial \xi} + A_n \frac{\partial X}{\partial \xi} + b'_n X \frac{\partial \vartheta}{\partial \eta} + B_n \frac{\partial X}{\partial \eta} + c_n X \frac{\partial \varphi}{\partial \eta} + F_n &= 0, \\ F_n &= f_n^{(1)} X \frac{\partial U'}{\partial \xi} + f_n^{(2)'} X \frac{\partial U'}{\partial \eta} + f_n^{(3)} X \frac{\partial V'}{\partial \eta} + f_n^{(4)} X^2 \frac{\partial A}{\partial \xi} + f_n^{(5)'} X^2 \frac{\partial A}{\partial \eta} + f_n^{(6)} X^2 \frac{\partial B}{\partial \eta} + f_n^{(7)}, \\ n &= 0, 1, \end{aligned}$$

$$a'_2 X \frac{\partial \vartheta}{\partial \xi} + b_2 X \frac{\partial \varphi}{\partial \xi} + B_2 \frac{\partial X}{\partial \xi} + c'_2 X \frac{\partial \varphi}{\partial \eta} + A'_2 \frac{\partial X}{\partial \eta} + F_2 = 0, \quad (14)$$

$$F_2 = f_2^{(1)'} X \frac{\partial U'}{\partial \xi} + f_2^{(2)} X \frac{\partial V'}{\partial \xi} + f_2^{(3)'} X \frac{\partial V'}{\partial \eta} + f_2^{(4)'} X^2 \frac{\partial A}{\partial \xi} + f_2^{(5)} X^2 \frac{\partial B}{\partial \xi} + f_2^{(6)'} X^2 \frac{\partial B}{\partial \eta} + f_2^{(7)'}$$

or, solving these three equations for $\partial \vartheta / \partial \xi$, $\partial \varphi / \partial \xi$, and $\partial X / \partial \xi$, we have

$$\begin{aligned} \frac{\partial \vartheta}{\partial \xi} &= \frac{K_1 A_0 - K_0 A_1}{X(a_0 A_1 - a_1 A_0)}, \\ \frac{\partial X}{\partial \xi} &= \frac{K_0 a_1 - K_1 a_0}{a_0 A_1 - a_1 A_0}, \\ \frac{\partial \varphi}{\partial \xi} &= \frac{1}{b_2 X(a_0 A_1 - a_1 A_0)} \left\{ a'_2 (K_0 A_1 - K_1 A_0) + B_2 (K_1 a_0 - K_0 a_1) + P_2 (a_1 A_0 - a_0 A_1) \right\}, \end{aligned} \quad (15-a)$$

where

$$\begin{aligned} K_n &= b'_n X \frac{\partial \vartheta}{\partial \eta} + B_n \frac{\partial X}{\partial \eta} + c_n X \frac{\partial \varphi}{\partial \eta} + F_n, \quad n=0,1, \\ P_2 &= c'_2 X \frac{\partial \varphi}{\partial \eta} + A'_2 \frac{\partial X}{\partial \eta} + F_2, \end{aligned} \quad (15-b)$$

and a_n , b_n , c_n , and $f_n^{(i)}$ are linear functions of θ , XA , and U' , and a'_n , b'_n , c'_n , and $f_n^{(i)'}$ linear ones of φ , XB , and V' . Further A_n are quadratics of θ , XA , and U' , and B_n are quadratics of products of previous linear ones.

The set of equations (14) or (15) is the final form of the set of moment equations, adequate to our numerical treatments. It is available for all cases of the three dimensional steady incompressible laminar boundary layer flow over a flat plate.

The numerical coefficients of these functions are tabulated in Table 1, 2, and 3.

4. The leading edge values

As the first application of the calculation scheme derived above, we take up a quasi two dimensional problem, that is the flow which is independent of the coordinate, η . Then, as may be seen from Eq. (15), once the values and derivatives of the unknown parameters θ , φ , and the boundary layer thickness parameter X at the leading edge of a flat plate are known, the values of these parameters can be evaluated integrating (15) along ξ -coordinate. We must therefore find so-called leading edge values.

Table. 1

	$\theta(\varphi)$	XA(XB)	U'(V')
a_0	0.300144×10^{-1}	-0.150794×10^{-1}	-0.253968×10^{-1}
a_1	0.158225×10^{-1}	-0.709957×10^{-2}	-0.103608×10^{-1}
a'_2	0.150072×10^{-1}	-0.753968×10^{-2}	0.373016×10^{-1}
b'_0	0.150072×10^{-1}	-0.753968×10^{-2}	-0.626984×10^{-1}
b'_1	0.541126×10^{-2}	-0.248918×10^{-2}	-0.280664×10^{-1}
b_2	0.150072×10^{-1}	-0.753968×10^{-2}	-0.626984×10^{-1}
c_0	0.150072×10^{-1}	-0.753968×10^{-2}	0.373016×10^{-1}
c_1	0.104113×10^{-1}	-0.461039×10^{-2}	0.177056×10^{-1}
c'_2	0.300144×10^{-1}	-0.150794×10^{-1}	-0.253968×10^{-1}
$f_0^{(1)}$	-0.125397×10^0	0.563492×10^{-1}	0.465079×10^0
$f_1^{(1)}$	-0.719047×10^{-1}	0.294048×10^{-1}	0.332302×10^0
$f_2^{(1) '}$	-0.626984×10^{-1}	0.281746×10^{-1}	-0.117460×10^0
$f_0^{(2) '}$	-0.626984×10^{-1}	0.281746×10^{-1}	0.582540×10^0
$f_1^{(2) '}$	-0.280664×10^{-1}	0.115079×10^{-1}	0.393651×10^0
$f_2^{(2)}$	-0.626984×10^{-1}	0.281746×10^{-1}	0.582540×10^0
$f_0^{(3)}$	-0.626984×10^{-1}	0.281746×10^{-1}	-0.117460×10^0
$f_1^{(3)}$	-0.438384×10^{-1}	0.178968×10^{-1}	-0.613492×10^{-1}
$f_2^{(3) '}$	-0.125397×10^0	0.563492×10^{-1}	0.465079×10^0
$f_0^{(4)}$	-0.150794×10^{-1}	0.793651×10^{-2}	0.634921×10^{-2}
$f_1^{(4)}$	-0.785714×10^{-2}	0.363095×10^{-2}	0.162698×10^{-2}
$f_2^{(4) '}$	-0.753968×10^{-2}	0.396825×10^{-2}	-0.218254×10^{-1}
$f_0^{(5) '}$	-0.753968×10^{-2}	0.396825×10^{-2}	0.281746×10^{-1}
$f_1^{(5) '}$	-0.248918×10^{-2}	0.119048×10^{-2}	0.115079×10^{-1}
$f_2^{(5)}$	-0.753968×10^{-2}	0.396825×10^{-2}	0.281746×10^{-1}
$f_0^{(6)}$	-0.753968×10^{-2}	0.396825×10^{-2}	-0.218254×10^{-1}
$f_1^{(6)}$	-0.536797×10^{-2}	0.244048×10^{-2}	-0.988095×10^{-2}
$f_2^{(6) '}$	-0.150794×10^{-1}	0.793651×10^{-2}	0.634921×10^{-2}
$f_0^{(7)}$	$-0.1 \times 10^{+1}$	$-0.5 \times 10^{+1}$	$0.2 \times 10^{+1}$
$f_1^{(7)}$	0.0	$-0.3 \times 10^{+1}$	$0.1 \times 10^{+1}$
$f_2^{(7) '}$	$-0.1 \times 10^{+1}$	$-0.5 \times 10^{+1}$	$0.2 \times 10^{+1}$

Table. 2

	A_0, A_2'	A_1
θ^2 (φ^2)	0.750361×10^{-2}	0.520563×10^{-2}
$\left(\frac{XA}{XB}\right)^2$	0.992063×10^{-2}	0.485119×10^{-2}
$\frac{\theta XA}{(\varphi XB)}$	-0.226190×10^{-1}	-0.128463×10^{-1}
$\frac{U'\theta}{(V'\varphi)}$	-0.126984×10^{-1}	-0.130664×10^{-1}
$\frac{U'XA}{(V'XB)}$	0.952381×10^{-2}	0.563492×10^{-2}
$\frac{U'^2}{(V'^2)}$	-0.587302×10^{-1}	-0.306746×10^{-1}

Table. 3

	B_0	B_1	B_2
$\theta\phi$	0.750361×10^{-2}	0.520563×10^{-2}	0.750361×10^{-2}
θXB	-0.113095×10^{-1}	-0.805195×10^{-2}	-0.113095×10^{-1}
ϕXA	-0.113095×10^{-1}	-0.479437×10^{-2}	-0.113095×10^{-1}
$X^2 AB$	0.992063×10^{-2}	0.485119×10^{-2}	0.992063×10^{-2}
$V\theta$	-0.313492×10^{-1}	-0.219192×10^{-1}	0.186508×10^{-1}
$U\phi$	0.186508×10^{-1}	0.885281×10^{-2}	-0.313492×10^{-1}
VXA	0.422619×10^{-1}	0.204563×10^{-1}	-0.327381×10^{-1}
UXB	-0.327381×10^{-1}	-0.148214×10^{-1}	0.422619×10^{-1}
UV	-0.587302×10^{-1}	-0.306746×10^{-1}	-0.587302×10^{-1}

Now as we have to assume that the boundary layer thickness vanishes at the leading edge, parameter X approaches to the limiting value zero when ξ tends to zero. Then for the right side of the first equation of Eq. (15) in order to tend to a finite value, the numerator has to tend to zero, and this condition gives a cubic equation of θ , one of whose solutions is obtained as the leading edge value of θ . By using this value of θ the limiting value of the right side gives the derivative $\partial\theta/\partial\xi$ at the leading edge.

In such a way, from the remainders of Eq. (15), the leading edge values of parameter ϕ , and derivatives $\partial\phi/\partial\xi$ and $\partial X/\partial\xi$ are determined. Once the leading edge values of parameters and their derivatives are known, Eq. (15) can be integrated by the fourth order Runge-Kutta method along ξ -coordinate.

5. The external flow

The main object of this first study is to analyse numerically three dimensional flow patterns near the separation region. The problem, therefore, will be restricted to the simplest case of the constant adverse pressure gradient. Further, as stated above, it is assumed that the flow is independent of η .

For this case, the equation of external inviscid flow may be written as

$$U' \frac{\partial U'}{\partial \xi} = -K_1, \quad U' \frac{\partial V'}{\partial \xi} = -K_2, \quad (16)$$

where constants K_1 and K_2 are given arbitrarily. By solving these equations, we obtain the velocity components of the external flow

$$U' = \sqrt{1-2K_1\xi}, \quad V' = V'_0 \left\{ 1 - \frac{K_2}{K_1 V'_0} (1 - \sqrt{1-2K_1\xi}) \right\} \quad (17)$$

where V'_0 is another arbitrary constant.

In numerical integration, assuming $K_1=1$, we may determine the external flow patterns by the values of V'_0 and K_2 .

In other cases of adverse pressure gradient we think that results are qualitatively the same as ours. An example we have tried is rather mathematical and calculations are mainly for the test of approximation⁶⁾:

$$U' = 1 - k_1\xi, \quad V' = V'_0 (1 - k_2\xi). \quad (18)$$

In this case, when V'_0 is zero, the boundary layer is that of Howarth, and when $U'=\text{const.}$, the problem is of crossflows discussed by Sowerby and Loos. In each case, our solutions are almost identical with them over all the layers, proving the accuracy of the moment-method. In the full case of (18), our results are all the same qualitatively, to the case of constant adverse pressure gradient, but the details are not mentioned here, because of the practical difficulty of realizing this field of outer flow.

6. Conclusion of the first case

Numerical calculations have been done in many cases corresponding to different sets

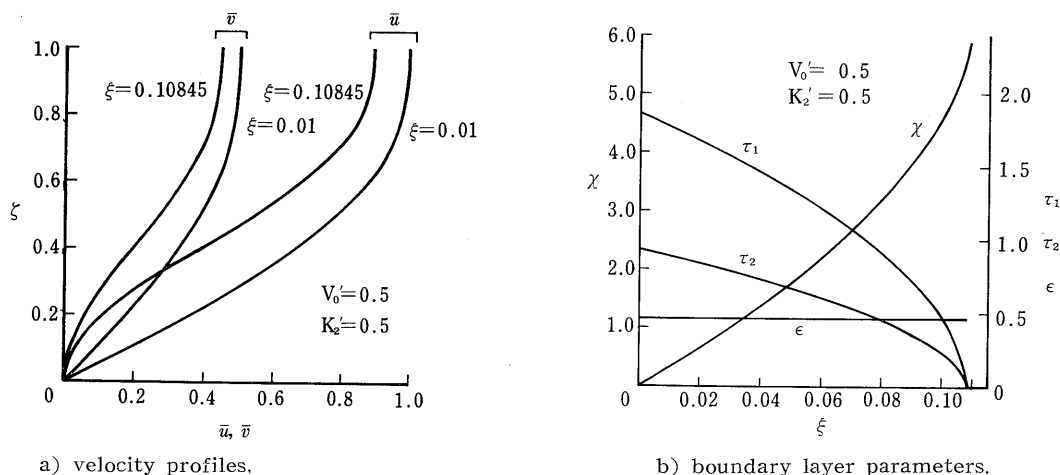


Fig. 1 Singular separation

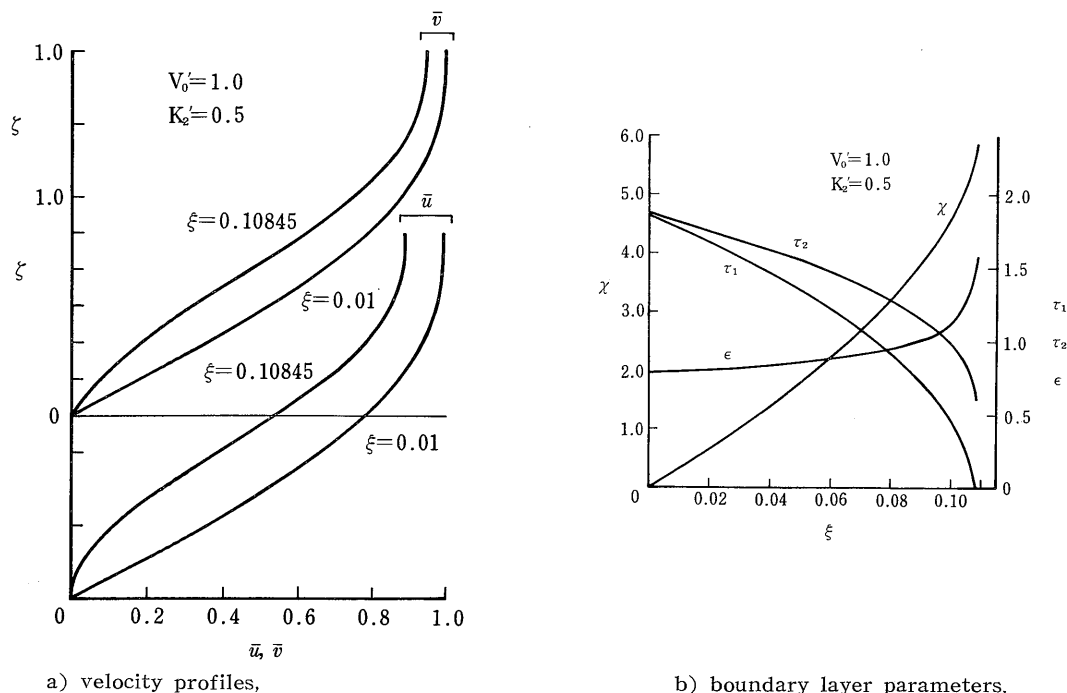
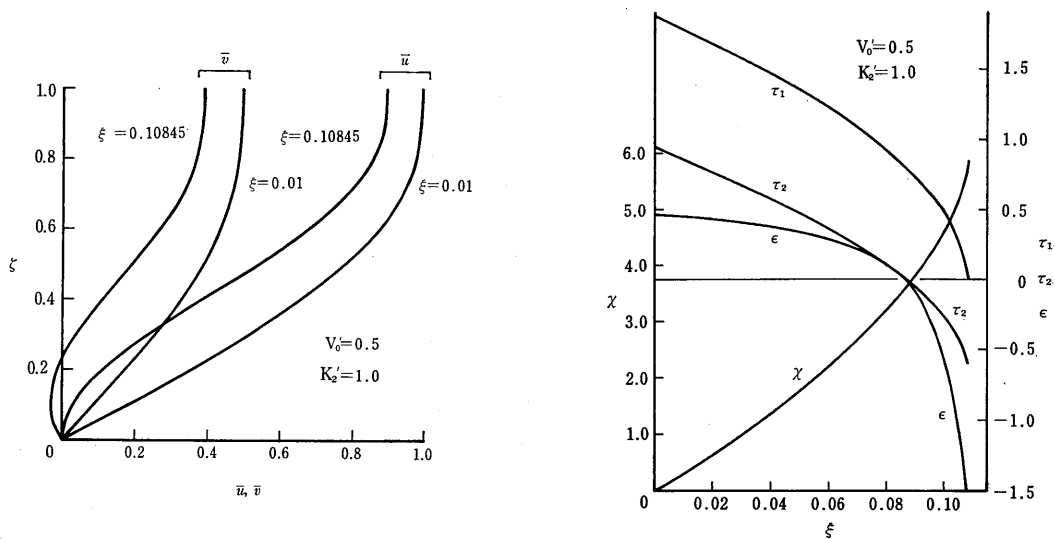
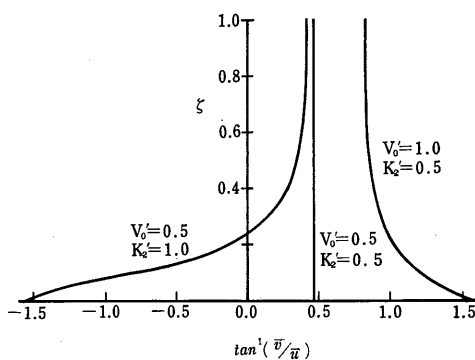
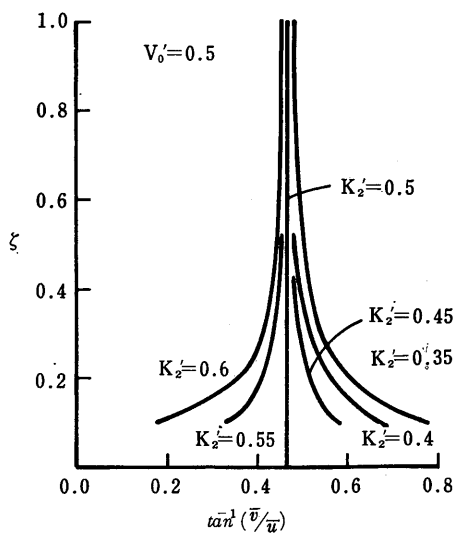


Fig. 2 Ordinary separation



a) velocity profiles,

b) boundary layer parameters.

Fig. 3 Ordinary separation**Fig. 4** The flow direction near the separation region.**Fig. 5** The flow direction near the separation region.

of V'_0 and K_2 . Results of calculation can be classified into three types with regard to the flow patterns near the separation region. When $V'_0 = K_2$, singular separation points occur as a line of such points (Fig. 1). When $V'_0 > K_2$, the direction of the secondary flow near the plate and that at the outer edge are the same (Fig. 2). When $V'_0 < K_2$, these two directions are opposite (Fig. 3). Figures show the change of the velocity profiles (Fig. ~a), and the development of the boundary layer thickness parameter X , the skin frictions $\tau_1 = (\partial \bar{u} / \partial \zeta)_{\zeta=0}$, $\tau_2 = (\partial \bar{v} / \partial \zeta)_{\zeta=0}$, and the direction of the limiting streamlines $\varepsilon = \tan^{-1}(\tau_2 / \tau_1)$ (Fig. ~b) along ξ -coordinate from the leading edge to the separation region.

As seen in Fig. 5, the difference between the flow directions of the ordinary separation and that of the singular one ($K_2 = 0.5$) seems to be not distinguished practically in $\zeta > 0.1$. This seems that, when the magnitude of the secondary flow is small, even though the ordinary separation occurs, the flow pattern near the separation region well resembles that of the singular separation. Horlock's statement that in many cases in turbines and compressors the flow is largely two dimensional up to the separation, may be, in a part of occasions at least, based on such resemblance.

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Errata (正誤表)

Page (ページ)	Column (列)	Line (行)	Should read (正)	Instead of (誤)
23	right(右)	17	layer	ayer
23		29	solutions	solutions
55		12	following	folllwing
66		11	Circuit	Circiut
117		20	$E_t - E_{25} = \dots\dots\dots$	$Et - E_{25} = \dots\dots\dots$
129		3	Noriyoshi MORITA*	Noriyoshi MORITA
133		21	Equilibrium	Fquilibrium