

Numerical Analysis of the Bending Moment in Inhomogeneous Thin Plates

Toshiyuki SAKATA*

(Received August 21, 1972)

The differential equation of the deflection surface of inhomogeneous thin plates whose moduli of elasticity vary with place, is derived and a numerical method useful for an accurate evaluation of the problem is proposed. The result of numerical calculations for uniformly loaded square plates whose Young's modulus varies with place, indicates that the maximum bending moment can be reduced, at least, to 70 per cent in comparison with that of the plate whose flexural rigidity is uniform in the case of a simply supported plate and to 53 per cent in the case of the plate whose two opposite sides are simply supported and the others are clamped.

1. Introduction

There are several papers described about the problem of an inhomogeneous media^{1,2)}, but the studies dealt with the bending problem of an inhomogeneous thin plate made of the material whose elastic properties vary with place are few. Evidently the distribution of the bending moment is not affected by the flexural rigidity when it is constant as in a homogeneous plate of uniform thickness. In the case of an inhomogeneous plate, however, the effect of the non-uniform flexural rigidity on the bending moment has not yet been clarified. The purpose of the present paper is to examine this ambiguous point.

First, the differential equation of the deflection surface of inhomogeneous thin plates is derived under the same assumptions as of homogeneous thin plates. Secondly, the differential equation is solved by the use of a double trigonometric series. Finally, the effect of inhomogeneity of the Young's modulus on the distribution of the bending moment is fully examined numerically for uniformly loaded square plates.

2. Differential equation of deflection surface

Taking the coordinate axes x and y in the middle plane of the plate and z axis perpendicular to that plane, the displacements u , v parallel to the x , y axes respectively are represented as in a homogeneous thin plate by

$$u = -z \frac{\partial w}{\partial x}, \quad v = -z \frac{\partial w}{\partial y} \quad (1)$$

where, w is the displacement parallel to the z axis. The relations between the stress and strain components for plane stresses in the x - y plane can be represented by the following equation

* Department of Mechanical Engineering

$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{22} & 0 \\ 0 & 0 & C_{33} \end{pmatrix} \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix} \quad (2)$$

where, C_{11} , C_{12} , C_{22} , C_{33} are the moduli of elasticity which are functions of x and y . Substituting Eqs. (1) and (2) into the equation of equilibrium

$$\left. \begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} &= 0 \\ \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} &= 0 \\ \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \sigma_z}{\partial z} &= 0 \end{aligned} \right\} \quad (3)$$

the differential equation of an inhomogeneous thin plate is obtained as

$$\begin{aligned} q = \frac{h^3}{12} \left[C_{11} \frac{\partial^4 w}{\partial x^4} + 2H \frac{\partial^4 w}{\partial x^2 \partial y^2} + C_{22} \frac{\partial^4 w}{\partial y^4} + 2 \left\{ \frac{\partial C_{11}}{\partial x} \frac{\partial^3 w}{\partial x^3} + \frac{\partial H}{\partial y} \frac{\partial^3 w}{\partial x^2 \partial y} \right. \right. \\ \left. \left. + \frac{\partial H}{\partial x} \frac{\partial^3 w}{\partial x \partial y^2} + \frac{\partial C_{22}}{\partial y} \frac{\partial^3 w}{\partial y^3} \right\} + \left(\frac{\partial^2 C_{11}}{\partial x^2} + \frac{\partial^2 C_{12}}{\partial y^2} \right) \frac{\partial^2 w}{\partial x^2} + \left(\frac{\partial^2 C_{12}}{\partial x^2} + \frac{\partial^2 C_{22}}{\partial y^2} \right) \frac{\partial^2 w}{\partial y^2} \right. \\ \left. + 4 \frac{\partial^2 C_{33}}{\partial x \partial y} \frac{\partial^2 w}{\partial x \partial y} \right] \quad (4) \end{aligned}$$

where, $H = C_{12} + 2C_{33}$, q : the lateral load acting on the upper surface of the plate, h : the thickness of the plate.

In the case of isotropy, Eq. (4) is expressed in the form containing only two elastic moduli, namely Young's modulus E and Poisson's ratio ν , as

$$\begin{aligned} q = \frac{h^3}{12} \left[C_{11} \nabla^4 w + 2 \frac{\partial C_{11}}{\partial x} \frac{\partial}{\partial x} (\nabla^2 w) + 2 \frac{\partial C_{11}}{\partial y} \frac{\partial}{\partial y} (\nabla^2 w) + \left\{ \frac{\partial^2 C_{11}}{\partial x^2} \right. \right. \\ \left. \left. + \frac{\partial^2}{\partial y^2} (\nu C_{11}) \right\} \frac{\partial^2 w}{\partial x^2} + \left\{ \frac{\partial^2}{\partial x^2} (\nu C_{11}) + \frac{\partial^2 C_{11}}{\partial y^2} \right\} \frac{\partial^2 w}{\partial y^2} \right. \\ \left. + 2 \frac{\partial^2}{\partial x \partial y} \left\{ (1-\nu) C_{11} \right\} \frac{\partial^2 w}{\partial x \partial y} \right] \quad (5) \end{aligned}$$

where, $C_{11} = E/(1-\nu^2)$

Assuming that Poisson's ratio ν is constant throughout the plate, the equation is reduced to

$$\begin{aligned} q = D_0 \left[e \nabla^4 w + 2 \frac{\partial e}{\partial x} \frac{\partial}{\partial x} (\nabla^2 w) + 2 \frac{\partial e}{\partial y} \frac{\partial}{\partial y} (\nabla^2 w) + \frac{\partial^2 e}{\partial x^2} \left(\frac{\partial^2 w}{\partial x^2} \right. \right. \\ \left. \left. + \nu \frac{\partial^2 w}{\partial y^2} \right) + \frac{\partial^2 e}{\partial y^2} \left(\nu \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) + 2(1-\nu) \frac{\partial^2 e}{\partial x \partial y} \frac{\partial^2 w}{\partial x \partial y} \right] \quad (6) \end{aligned}$$

where, $D_0 = E_0 h^3 / 12(1-\nu^2)$, $e = E/E_0$, E_0 : the averaged value of E throughout the plate.

3. Numerical method solving the differential equation

As a solution of the differential equation (6), the deflection w can be represented as

$$w = f(x, y) \sum_{k=1}^K \sum_{l=1}^L w_{kl} \sin \frac{k\pi}{a} x \sin \frac{l\pi}{b} y \quad (7)$$

where, a , b are the side lengths of a rectangular plate.

Substituting w into Eq. (6), it becomes

$$q/D_0 = \sum_{k=1}^K \sum_{l=1}^L w_{kl} E_{kl} \quad (8)$$

where,

$$E_{kl} = C_{kl}^{SS} \sin \frac{k\pi}{a} x \sin \frac{l\pi}{b} y + C_{kl}^{CS} \cos \frac{k\pi}{a} x \sin \frac{l\pi}{b} y \\ + C_{kl}^{SC} \sin \frac{k\pi}{a} x \cos \frac{l\pi}{b} y + C_{kl}^{CC} \cos \frac{k\pi}{a} x \cos \frac{l\pi}{b} y \quad (9)$$

$$C_{kl}^{SS} = e \left[\nabla^4 f - \left\{ 6 \left(\frac{k\pi}{a} \right)^2 + 2 \left(\frac{l\pi}{b} \right)^2 \right\} \frac{\partial^2 f}{\partial x^2} - \left\{ 6 \left(\frac{l\pi}{b} \right)^2 + 2 \left(\frac{k\pi}{a} \right)^2 \right\} \frac{\partial^2 f}{\partial y^2} \right. \\ \left. + \left\{ \left(\frac{k\pi}{a} \right)^2 + \left(\frac{l\pi}{b} \right)^2 \right\}^2 f \right] \\ + 2 \frac{\partial e}{\partial x} \frac{\partial}{\partial x} \left[\nabla^2 f - \left\{ 3 \left(\frac{k\pi}{a} \right)^2 + \left(\frac{l\pi}{b} \right)^2 \right\} f \right] \\ + 2 \frac{\partial e}{\partial y} \frac{\partial}{\partial y} \left[\nabla^2 f - \left\{ \left(\frac{k\pi}{a} \right)^2 + 3 \left(\frac{l\pi}{b} \right)^2 \right\} f \right] \\ + \frac{\partial^2 e}{\partial x^2} \left[\frac{\partial^2 f}{\partial x^2} + \nu \frac{\partial^2 f}{\partial y^2} - \left\{ \left(\frac{k\pi}{a} \right)^2 + \nu \left(\frac{l\pi}{b} \right)^2 \right\} f \right] \\ + \frac{\partial^2 e}{\partial y^2} \left[\frac{\partial^2 f}{\partial y^2} + \nu \frac{\partial^2 f}{\partial x^2} - \left\{ \nu \left(\frac{k\pi}{a} \right)^2 + \left(\frac{l\pi}{b} \right)^2 \right\} f \right] \\ + 2(1-\nu) \frac{\partial^2 e}{\partial x \partial y} \frac{\partial^2 f}{\partial x \partial y} \\ C_{kl}^{SC} = 4e \frac{l\pi}{b} \frac{\partial}{\partial y} \left[\nabla^2 f - \left\{ \left(\frac{k\pi}{a} \right)^2 + \left(\frac{l\pi}{b} \right)^2 \right\} f \right] \\ + 4 \frac{\partial e}{\partial x} \frac{l\pi}{b} \frac{\partial^2 f}{\partial x \partial y} + 2 \frac{\partial e}{\partial y} \frac{l\pi}{b} \left[\nabla^2 f + 2 \frac{\partial^2 f}{\partial y^2} \right. \\ \left. - \left\{ \left(\frac{k\pi}{a} \right)^2 + \left(\frac{l\pi}{b} \right)^2 \right\} f \right] + \frac{\partial^2 e}{\partial x^2} \left[2\nu \frac{l\pi}{b} \frac{\partial f}{\partial y} \right] \\ + \frac{\partial^2 e}{\partial y^2} \left[2 \frac{l\pi}{b} \frac{\partial f}{\partial y} \right] + 2(1-\nu) \frac{\partial^2 e}{\partial x \partial y} \frac{l\pi}{b} \frac{\partial f}{\partial x} \\ C_{kl}^{CS} = 4e \frac{k\pi}{a} \frac{\partial}{\partial x} \left[\nabla^2 f - \left\{ \left(\frac{k\pi}{a} \right)^2 + \left(\frac{l\pi}{b} \right)^2 \right\} f \right] + 4 \frac{\partial e}{\partial y} \frac{k\pi}{a} \frac{\partial^2 f}{\partial x \partial y} \\ + 2 \frac{\partial e}{\partial x} \frac{k\pi}{a} \left[\nabla^2 f + 2 \frac{\partial^2 f}{\partial x^2} - \left\{ \left(\frac{k\pi}{a} \right)^2 + \left(\frac{l\pi}{b} \right)^2 \right\} f \right] \\ + \frac{\partial^2 e}{\partial x^2} \left[2 \frac{k\pi}{a} \frac{\partial f}{\partial x} \right] + \frac{\partial^2 e}{\partial y^2} \left[2\nu \frac{k\pi}{a} \frac{\partial f}{\partial y} \right] \\ + 2(1-\nu) \frac{\partial^2 e}{\partial x \partial y} \frac{k\pi}{a} \frac{\partial f}{\partial y} \\ C_{kl}^{CC} = 8e \frac{k\pi}{a} \frac{l\pi}{b} \frac{\partial^2 f}{\partial x \partial y} + 4 \frac{\partial e}{\partial x} \frac{k\pi}{a} \frac{l\pi}{b} \frac{\partial f}{\partial y} \\ + 4 \frac{\partial e}{\partial y} \frac{k\pi}{a} \frac{l\pi}{b} \frac{\partial f}{\partial x} + 2(1-\nu) \frac{k\pi}{a} \frac{l\pi}{b} \frac{\partial^2 e}{\partial x \partial y} \cdot f \quad (10)$$

E_{kl} can be expressed by the use of a double trigonometric series as

$$E_{kl} = \sum_{m=1}^K \sum_{n=1}^L B_{kl}^{mn} \sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y \quad (11)$$

where,

$$B_{kl}^{mn} = \frac{4}{ab} \int_0^a \int_0^b E_{kl} \sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y dx dy \quad (12)$$

q/D_0 can be expressed as

$$\frac{q}{D_0} = \sum_{m=1}^K \sum_{n=1}^L q_{mn} \sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y \quad (13)$$

where,

$$q_{mn} = -\frac{4}{ab} \int_0^a \int_0^b \frac{q}{D_0} \sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y dx dy$$

From Eqs. (8), (11) and (13), we obtaine

$$q_{mn} = \sum_{k=1}^K \sum_{l=1}^L w_{kl} B_{kl}^{mn} \quad (14)$$

where,

$$m=1, 2, \dots, K, \quad n=1, 2, \dots, L$$

The unknown coefficients w_{kl} can be obtained by solving the simultaneous equations (14). The function $f(x, y)$ is determined so as the deflection w satisfies the prescribed boundary conditions.

4. Numerical results

Numerical calculations were performed for a square plate with side length a as shown in Fig.1. The lateral load q is assumed uniform q_0 and the Young's modulus E varies only in the x -direction by

$$E = E_0 (1 + \alpha \cos \frac{2m}{a} \pi x) \quad (15)$$

where, α : constant, m : integer.

Assuming the Young's modulus E as in Eq. (15), the constant B_{kl}^{mn} in Eq. (12) vanishes when n is not equal to l .

Accordingly, the unknowns w_{kl} can be determined by solving L times the simultaneous equations of K unknowns.

a) The square plate whose two opposite sides ($x=0, a$) are clamped and the others ($y=0, a$) are supported.

In this case the boundary conditions are satisfied when $f(x, y)$ is expressed as

$$f(x, y) = \sin \frac{\pi}{a} x \quad (16)$$

Table 1 The accuracy of the calculating method used in this paper.

	$\frac{w_{max}}{10^{-2} q_0 a^4 / D_0}$	$\frac{M_{xmax}}{10^{-1} q_0 a^2}$	$\frac{M_{ymax}}{10^{-1} q_0 a^2}$
Calculated values	0.1918	-0.6962	0.2432
Exact values ³⁾	0.192	-0.697	0.244

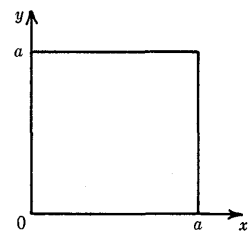


Fig. 1 A shape of a plate and x, y coordinates.

Table 2 Examples of the convergency of series

$K=L$	$\frac{w_{max}}{10^{-2} q_0 a^4 / D_0}$	$\frac{M_{xmax}}{10^{-1} q_0 a^2}$	$\frac{M_{ymax}}{10^{-1} q_0 a^2}$
20	0.20597	-0.54329	0.35783
40	0.20479	-0.54555	0.36095
60	0.20457	-0.54547	0.36183
80	0.20449	-0.54504	0.36222
x	$0.5a$	$0, a$	$0.5a$

$$\alpha = -0.5, \quad y = 0.5b \quad e = 1 + \alpha \cos \frac{4\pi}{a} x$$

To examine the accuracy of the numerical method proposed, a calculation was performed for the particular case when α is zero, namely the flexural rigidity is unifom. The results are shown in Table 1. The results given in the table show that the numerical error is less than 1 per cent. The bending moment becomes minimum at the middle point of the

sides ($x=0, a$) and maximum at the center of the plate. In order to reduce the flexural rigidity of the plate at those peak points, m is assumed 2 and α negative. As an example showing the convergency of the series, numerical results are shown in Table 2 when α is -0.5 . From the tables it can be concluded that the series used in the calculation converges rapidly and the results are sufficiently accurate. The absolute value of the maximum moment M_{xmax} (in the direction of x axis) and M_{ymax} (in the direction of y axis) are shown by curves in Fig. 2 as the function of α . The figure shows that the absolute value of the maximum bending moment becomes minimum when α is -0.86 and its magnitude is reduced to 57 per cent in comparison with that in the homogeneous plate.

b) A simply supported square plate.

When $f(x, y)$ is given as

$$f(x, y) = 1 \quad (17)$$

the boundary conditions are satisfied. In this case we take m equal to 1 and α positive. The maximum value of the bending moment becomes minimum when α is 0.7 and the value is reduced to 70 per cent in comparison with that in the homogeneous plate as shown in Fig. 3. When E is the function of x and y the maximum value of the bending moment can be reduced furthermore. But the numerical calculation becomes more laborious since B_{kl}^{mn} does not vanish even if n is not equal to l .

5. Conclusion

The differential equation of the deflection surface of an inhomogeneous thin plate was derived and solved by the use of a double trigonometric series. The numerical results show that the accuracy of the method proposed herein is sufficiently good and the numerical error is less than 1 per cent and the convergency of the series is rapid enough for practical use. According to the numerical results for uniformly loaded square plates, it is found that the maximum bending moment occurring in inhomogeneous plate can be reduced considerably in comparison with that

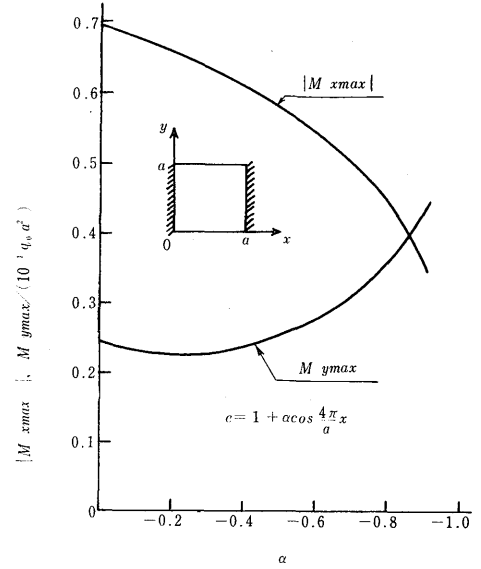


Fig. 2 The maximum value of the bending moment of a uniformly loaded square plate whose opposite sides are clamped and the others are simply supported.

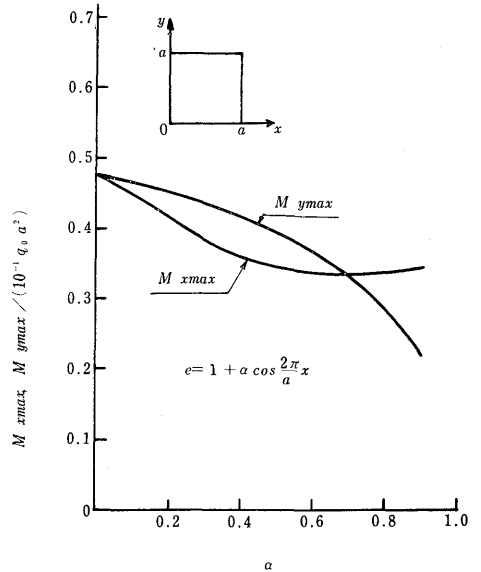


Fig. 3 The maximum value of the bending moment of a uniformly loaded and simply supported plate.

occurs in homogeneous plate when Young's modulus varies in one direction only.

The result, however, does not indicate that an inhomogeneous plate is better than a homogeneous plate because the plate having the prescribed elastic property cannot be available always, and furthermore the allowable stress at the peak point affects immediately on the strength of the plates.

Nevertheless, those results obtained here are of theoretical interest from the stand point of strength of materials.

Acknowledgement

The author would like to express Prof. Dr. H. Okubo for helpful suggestions.

References

- | | |
|--|--|
| 1) H. Okubo and S. Seika, Trans. JSME, 28-194(1962), 1293. | 3) S.P.Timoshenko and S.Woinowsky-Krieger, "Theory of Plates and Shells", 2nd. ed. (1959), McGraw-Hill, 187. |
| 2) R. Furuhashi and M. Kataoka, Trans. JSME, 33-253(1967), 1331. | |