

Ray Tracing of Whistler Waves

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Authors consider collisional effects on whistler ray paths which are essential to the phenomenon of the penetration of whistler waves through the lower boundary of the ionosphere. An investigation is made of the ray equations suitable for inhomogeneous waves.

Some ray paths are numerically computed in the case of "penetration" above mentioned, by means of our formulation.

1. Introduction

In the last ten years, many experimental data have been obtained as to the propagation of VLF waves such as "whistler-mode". Calculations of whistler ray paths are needed in order to understand these experimental results and we can use a set of differential equations for ray tracing suitable for inhomogeneous, anisotropic medium, which has been already established in some references.^{1),2)} We are interested in the investigation by means of the given procedure, how the ray paths vary with frequency, latitude, initial wave normal angle, electron density profile and other variables. This continuative work is still now important and provides a basis for comparison of the theory with the experiment, which was first attempted by I. Yabroff.³⁾

Their formulation is, however, based on the assumption that effects of the collision between electrons and neutral particles are neglected, or negligibly small. It can not be immediately applied to the case in which the collisional effects are dominant. We should consider these effects in the neighborhood of the lower boundary of the ionosphere where the electromagnetic wave can not propagate without the collisional effects, i.e. "cut-off". These effects may not be so effective to the total characteristic of ray paths and the value of the dispersion of whistler waves observed, as to the phenomenon of the penetration of whistler waves through the lower boundary of the ionosphere.

The present work is to develop the ray equations suitable for inhomogeneous waves, the direction of whose wave normal is expressed by the complex angle. The reason we must treat waves of this kind is just the result of the facts that Snell's law is satisfied and that the refractive index is complex number because of finite collisional effects. The intension of our treatment is equivalent to that of Booker quartic. We take a model of the flat ionosphere which is appropriate to describe the local region of the lower boundary of the ionosphere, but the extension of the procedure

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to the case of a curved ionosphere may be straight forward. In section 2, some properties of inhomogeneous waves are illustrated and section 3 is devoted to describe our procedure of ray tracing. An example of the results of numerical computation is shown.

2. Wave Normal of Inhomogeneous Waves

Considering a straight sharp boundary between homogeneous plasmas and vacuum with no static magnetic field, the refractive index $\mathbf{M}$ is in general written as

$$\mathbf{M} = (\mu_0 - i\chi_0)(\sin\theta_0 \mathbf{i} + \cos\theta_0 \mathbf{k}), \quad (1)$$

in plasmas, where μ_0, χ_0 are real numbers and $\mathbf{i}, \mathbf{k}$ are unit vectors and the direction of the stratification of a plasma density profile is expressed as x -axis, as shown in Fig. 1. The condition that Snell's law as follows,

$$(\mu_0 - i\chi_0)\sin\theta_0 = \sin\theta_i, \quad (2)$$

is to be satisfied, requires θ_0 to become complex number, where θ_i is the real angle of the incidence of the wave incident on the plasma.

Some field quantity A is expressed as,

$$\begin{aligned} A &\propto \exp[-ik\mathbf{M}\cdot\mathbf{r}] \\ &= \exp[-ik\{(\mu_0 - i\chi_0)\sin\theta_0 x + (\mu_0 - i\chi_0)z\cos\theta_0\}] \\ &= \exp[-ikx\sin\theta_i] \cdot \exp[-ik(\mu_0 - i\chi_0)z\cos\theta_0], \end{aligned} \quad (3)$$

where k is the wave number in vacuum which depends on wave frequency. We can conveniently express the refractive index, $\mathbf{M}$, in another form as,

$$\mathbf{M} = \mu - i\chi, \quad (1')$$

where χ is demanded to be taken toward z -direction by Snell's law.

Snell's law is given as,

$$\mu\sin\theta = \sin\theta_i. \quad (2')$$

Field quantity is now given as,

$$\begin{aligned} A &\propto \exp(-ik\mu\cdot\mathbf{r}) \cdot \exp(-k\chi\cdot\mathbf{r}) \\ &= \exp\{-ik(\mu x\sin\theta + \mu z\cos\theta)\} \cdot \exp(-k\chi z) \\ &= \exp(-ikx\sin\theta_i) \cdot \exp(-ik\mu z\cos\theta) \cdot \exp(-k\chi z), \end{aligned} \quad (3')$$

where the angle θ , is real number in Fig. 1. The factor $\{-ik(\mu_0 - i\chi_0)z\cos\theta_0\}$, in Eq. (3) is easily rewritten as $\{-ik\mu z\cos\theta - k\chi z\}$ in Eq. (3') as follows,

$$\begin{aligned} (\mu_0 - i\chi_0)\cos\theta_0 &= \sqrt{(\mu_0 - i\chi_0)^2 - (\mu_0 - i\chi_0)^2\sin^2\theta_0} \\ &= \sqrt{M^2 - \sin^2\theta_i} \\ &= \sqrt{(\mu^2 - \chi^2 - 2i\mu\chi\cos\theta) - \mu^2\sin^2\theta} \\ &= \sqrt{\mu^2\cos^2\theta - 2i\mu\chi\cos\theta - \chi^2} \\ &= \mu\cos\theta - i\chi. \end{aligned} \quad (4)$$

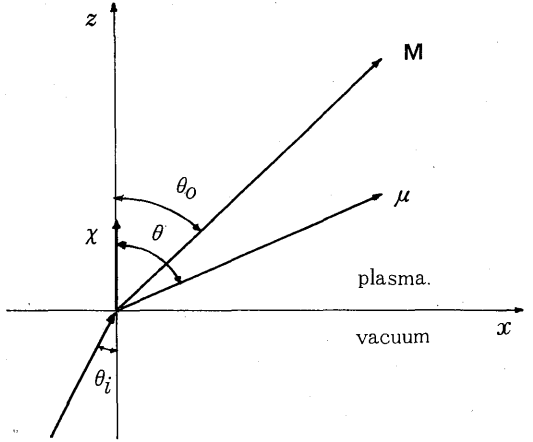


Fig. 1 The vectors $\mu, \chi, \mathbf{M}$ in Eqs.(1) and (1') are shown in the coordinate system.

Eq.(3) is equivalent to Eq.(3'), and we may understand the direction of the wave normal of inhomogeneous waves to be expressed with real angle θ , rather than complex angle θ_0 . The refractive index $\mathbf{M}$, is obtained by solving the dispersion relation in the alternative form such as Eq.(1') in the following.

3. Procedure of Ray Tracing

The wave packet is generally expressed as follows,

$$a(\mathbf{r}, t) = \int A(\mathbf{k}', \omega) \exp\{i(\omega t - \mathbf{k}' \cdot \mathbf{r})\} d\omega d\mathbf{k}'. \quad (5)$$

Integrating for the variable of $\mathbf{k}'$, we should take up the solution of the dispersion relation for a given ω as,

$$\mathbf{k}' = \mathbf{k}'(\omega). \quad (6)$$

We put $\mathbf{k}'$ as follows,

$$\mathbf{k}' = k\mathbf{M} = k(\mu - i\chi), \quad (7)$$

where the terms, μ , χ express the behavior of the wave normal and the damping effect of waves, respectively. The position of the wave packet, or the position of ray is found from the condition that the exponent in Eq.(5) must be stationary with respect to variations of the direction of μ and the wave number k (the well-known principle of stationary phase). Hereafter, we can take the same procedure as given in the reference,¹⁾ except that we take $\mu = |\mu|$ given in the reference for μ which we have got in Eq.(7). A set of differential equations for ray tracing is given as follows,

$$\left. \begin{aligned} \frac{dx}{dt} &= \frac{c}{\mu^2} \left(\mu \sin \theta - \cos \theta \frac{\partial \mu}{\partial \theta} \right), \\ \frac{dz}{dt} &= \frac{c}{\mu^2} \left(\mu \cos \theta + \sin \theta \frac{\partial \mu}{\partial \theta} \right), \\ \frac{d\theta}{dt} &= -\frac{c}{\mu^2} \sin \theta \frac{\partial \mu}{\partial z}, \end{aligned} \right\} \quad (8)$$

in the two dimension. The coordinate system taken in the above equation is shown in Fig. 2. It is noted that ψ (angle between $\mathbf{M}$ and $\mathbf{Y}$) is complex number. This fact should be paid attention to when the quantity μ and its derivatives which are necessary for the righthand side of Eq.(8), are calculated from the Appleton-Hartree formula. These quantities are obtained from the following simultaneous equations.

$$(\mu^2 - \chi^2 - 2i\mu\chi \cos \theta) = M^2(\psi), \quad (9)$$

$$[(\mu \sin \theta, 0, \mu \cos \theta) - i(0, 0, \chi)]^2 = M^2(\psi), \quad (9')$$

$$\cos \psi = \frac{\mu \cos(\theta - \phi) - i\chi \cos \phi}{\sqrt{\mu^2 - \chi^2 - 2i\mu\chi \cos \theta}}, \quad (10)$$

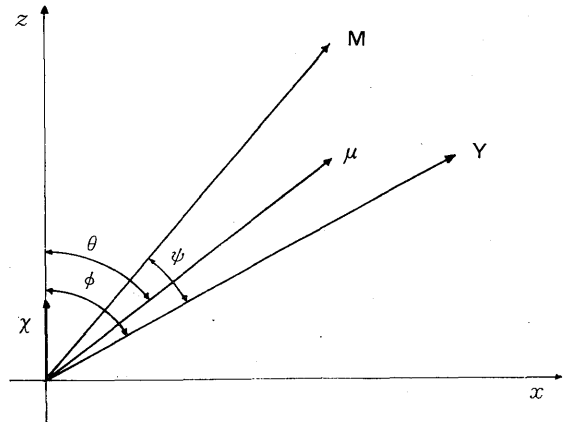


Fig. 2 The coordinate system and some variables in Eqs.(8).

$$M^2(\psi) = 1 - \frac{2X(1-X-iZ)}{D}, \quad (11)$$

$$D = 2(1-X-iZ)(1-iZ) - Y^2 \sin^2 \psi + S, \quad (12)$$

$$S = \pm \sqrt{Y^4 \sin^4 \psi + 4Y^2(1-X-iZ)^2 \cos^2 \psi}, \quad (13)$$

$$\frac{\partial \mu}{\partial \theta} = \frac{E_r C_i - E_i C_r}{B_r C_i - B_i C_r}, \quad (14)$$

$$\frac{\partial \mu}{\partial z} = \frac{F_r C_i - F_i C_r}{B_r C_i - B_i C_r}, \quad (15)$$

$$B \equiv B_r + iB_i = A(\mu - iX \cos \theta) - M \frac{\partial M}{\partial \psi} \cos(\theta - \phi), \quad (16)$$

$$C \equiv C_r + iC_i = -A(X + i\mu \cos \theta) + iM \frac{\partial M}{\partial \psi} \cos \phi, \quad (17)$$

$$E \equiv E_r + iE_i = -\mu [A(iX \sin \theta) + M \frac{\partial M}{\partial \psi} \sin(\theta - \phi)], \quad (18)$$

$$F \equiv F_r + iF_i = -M^2 \sin \psi \left(\frac{\partial M}{\partial X} \frac{\partial X}{\partial z} + \frac{\partial M}{\partial Z} \frac{\partial Z}{\partial z} \right), \quad (19)$$

$$A \equiv A_r + iA_i = \frac{\partial M}{\partial \psi} \cos \psi - M \sin \psi, \quad (20)$$

$$\frac{\partial M}{\partial \psi} = \frac{(M^2 - 1) Y^2 \sin \psi \cos \psi}{MD} \left[1 - \frac{1}{S} \{ Y^2 \sin^2 \psi - 2(1-X-iZ)^2 \} \right], \quad (21)$$

$$\frac{\partial M}{\partial X} = \frac{1}{MD} \left[2X - 1 + iZ + (M^2 - 1) \left\{ 1 - iZ + \frac{2}{S} Y^2 (1-X-iZ) \cos^2 \psi \right\} \right], \quad (22)$$

$$\frac{\partial M}{\partial Z} = \frac{i}{MD} \left[X + (M^2 - 1) \left\{ 2 - X - 2iZ + \frac{2}{S} Y^2 (1-X-iZ) \cos^2 \psi \right\} \right], \quad (23)$$

where

X = normalized electron density = $(Ne^2)/(\omega^2 m \epsilon_0)$,

Y = normalized magnitude of the earth's magnetic field vector $\mathbf{Y} = (e\mathbf{B})/(\omega m)$,

Z = normalized collision frequency = ν_0/ω ,

ψ = angle between ρ and $Y = \cos^{-1}(\rho_r Y_r + \rho_\theta Y_\theta + \rho_\phi Y_\phi)/(\mu Y)$,

N = electron density,

e = charge on an electron,

m = mass of an electron,

ω = angular wave frequency = $2\pi f$

ϵ_0 = dielectric constant of free space,

ν_0 = collision frequency in collisions per second.

Angles θ , ϕ and ψ are shown in Fig. 2.

The reason why we put the refractive index M , in the rather simple form like Eq.(9') is that our purpose is to find the collisional effect on the coupling between the waves in the ionosphere and those in the atmosphere, where the direction of the local stratification of an electron density profile in the lower boundary of the ionosphere may be taken to be constant, x -direction, or X is taken to have a constant direction, z -axis. $\mathbf{Y}$ is taken to be locally constant by the same reason. Eq. (8)-(23) can only be applied to the ray paths in the magnetic meridian plane and the refractive index M in the form like Eq.(9') has a real x -component which is only applied to the case of the wave propagation being coupled with the electromagnetic waves in the

atmosphere below the ionosphere. When the waves incident from above on the lower boundary of the ionosphere do not penetrate into the atmosphere, the x -component of the refractive index M cannot be real value.

A solution of the dispersion relation, Eq. (9) and (10) is shown in Fig.3 for the following parameters, $X=100.0$ (number density of electrons $N=30.0/\text{cc}$), $Y=100.0$ (magnitude of the earth's magnetic field $H=0.2$ gauss), $Z=10.0$ (frequency of collision of electrons with heavy particles $\nu=3.0 \times 10^5$ c/s), where wave frequency $f=5$ kc/s. These parameters are about typical values for the ionosphere at the height of several ten kilometers over the ground, the region of which are under the discussion. $\phi=0.0^\circ$, 30.0° , 45.0° and 60.0° is corresponding to the magnetic latitude of 90.0° , 41.0° , 26.5° and 16.0° , respectively. The broken line denotes the solution of homogeneous waves for the same parameters, in which ψ is real.

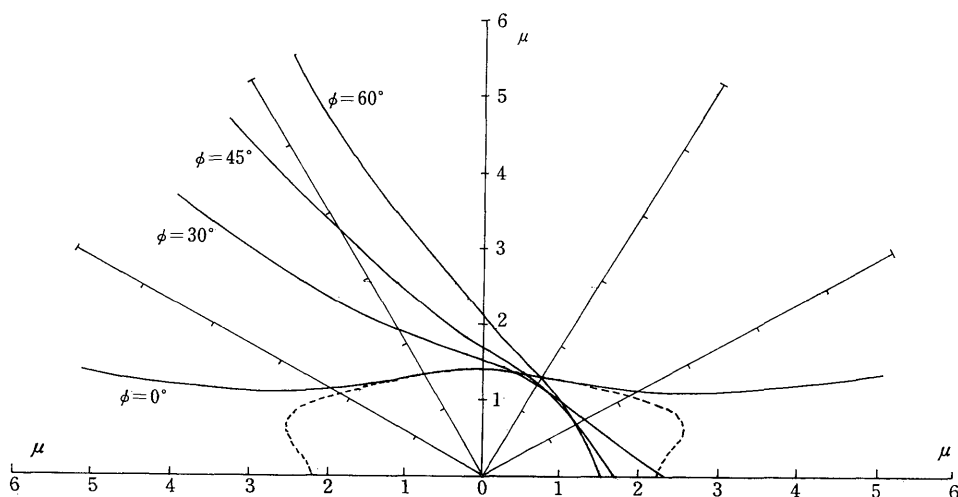


Fig. 3 Solutions of Eqs.(9) and (10) for $X=100.0$, $Y=100.0$, $Z=10.0$, $\phi=0.0^\circ$, 30.0° , 45.0° and 60.0° .

An example of the computations of whistler ray paths is shown in Fig. 4. Parameters of the medium at the starting point of the ray paths which is at the height of 70.0 km over the ground, are $X=100.0$, $Y=100.0$, $Z=10.0$, $\phi=45.0^\circ$ and initial wave normal angles θ are taken to be 180.0° , 200.0° , 215.0° , 225.0° , and 230.0° . These parameters are corresponding to the whistler waves arrived at the Kasugai Observatory near Nagoya City from the Southern Hemisphere. A model of an electron density profile and frequency of collision is as follows,

$$N \propto (z-40.0)^2 \quad \text{for } z \geq 40.0,$$

$$N = 0 \quad \text{for } z \leq 40.0,$$

$$\nu \propto \exp(-z/10.0),$$

where z is the height over the ground in km.

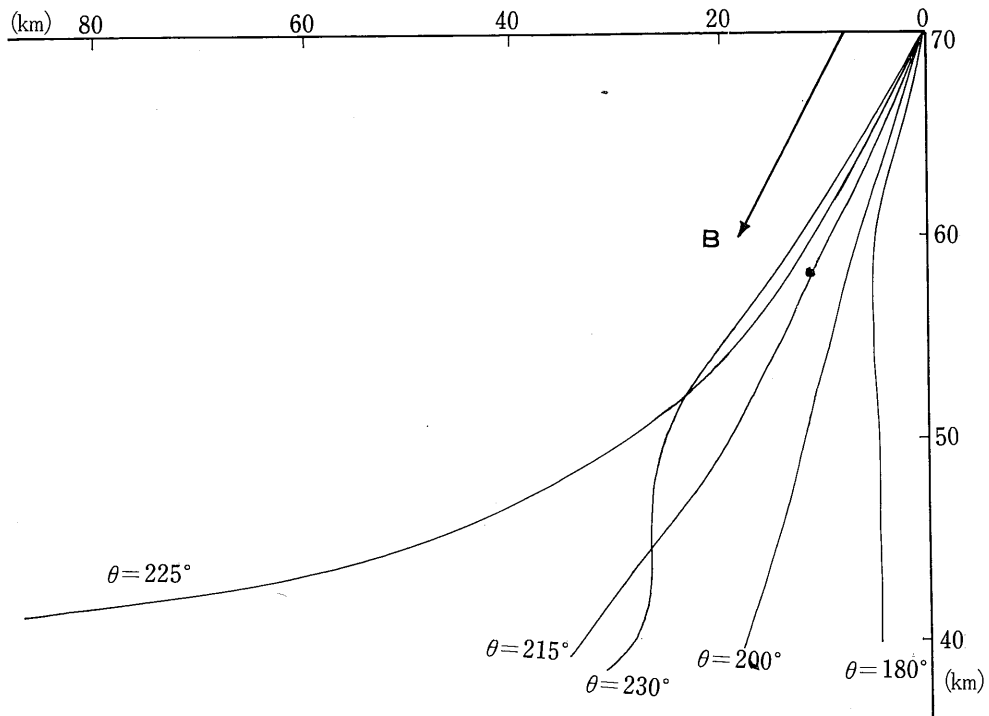


Fig. 4 Whistler ray paths starting from the ionosphere at the height of 70.0 km over the ground ("penetration" of E.M. waves from the ionosphere into the atmosphere).

4. Conclusion and Discussion

It can be concluded by our several results of numerical computations that as the direction of initial wave normal approaches to the direction of the earth's magnetic field, the horizontal distance of the arrival at the lowest boundary of the ionosphere from the starting point becomes longer. However, in order to derive a general rule of ray paths of inhomogeneous waves, more numerical computations are to be carried out.

Our formulation has several restrictions described in section 3, which is not able to be applied to the work of tracing the total ray paths in the curved ionosphere. In such a case, we must solve the dispersion relation like Eq.(9) and (10) by considering that the projection of $\mathbf{M} = \mu - i\chi$ in Eq.(7) on the direction of local stratification is to be conserved (Snell's law). A set of Eqs.(8) dominates the variation of μ , that is, as far as the real part of $\mathbf{M}$, Snell's law is automatically satisfied by Eqs.(8), but it should be noted that the tangential component of the imaginary part of $\mathbf{M}$ on each boundary is also conserved through all the steps.

The differential equations (8) is solved by the method of the Runge-Kutta-Gill process. The use of a finite step size means that the curve is being replaced by a series of straight line segments. This can lead to errors known as truncation errors, and to reduce that the step size should be as small as possible. However, it is not appropriate to use too small a step not only because of a too long time of computations

but also because of a finite effective figure of the real machine. We can have the integration curve becoming definite before using such a small step size, except for the use of the electron density profiles which are particularly steep.

Numerical computations are made on Facom 270 - 30.

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